MATH HANDOUT #2

All the Calculus You'll Ever Need

Roughly speaking, calculus is all about how to find the *slopes* of lines & curves.

Functions & Curves

Generally speaking, let a curve be represented by a function $Y = f(X)$, where the dependent variable is Y (value on the vertical *y*-axis) and the independent variable is X (value on the horizontal *x*-axis) and f is some elaborate relationship between them ('the curve').

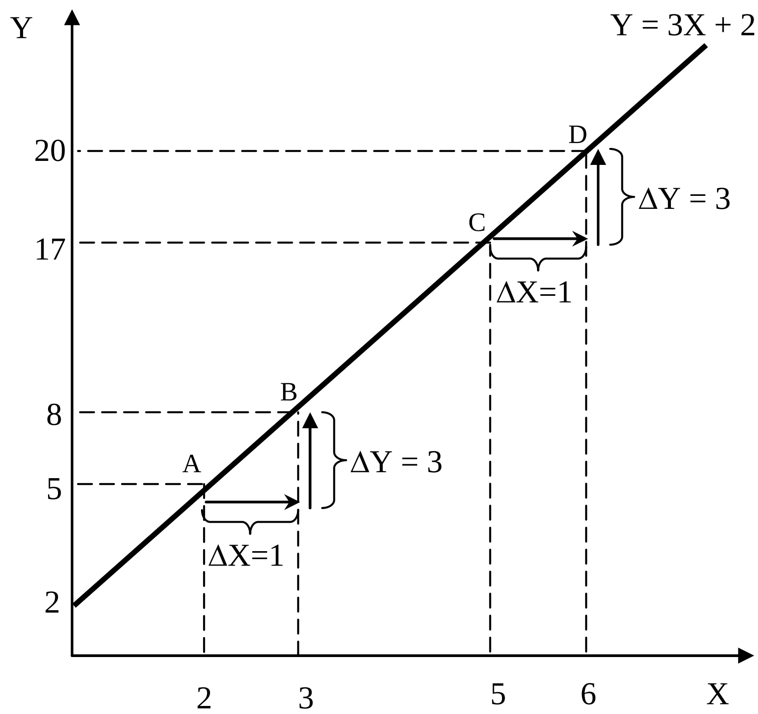
As mentioned in the earlier notes, this relationship is a simple "if X, then Y " statement.

For instance, $Y = f(X)$ could be:

e.g $Y = 3X + 2$

so:
if X = 0, then Y = 2
if X = 1, then Y = 5,
if X = 2, then Y = 8
if X = 3, then Y = 11
if X = 4, then Y = 14
if X = 5 then Y = 17
if X = 6, then Y = 20

and so on. And plotted out will look something like the following:



Now, you know that the slope is found simply by the rule "rise over run". Let the Greek letter Δ denote "change in". So, ΔY = change in Y ("rise") and ΔX = change in X ("run"). Thus slope ("rise over run") is:

Slope = rise/run = (change in Y)/(change in X) = $\Delta Y/\Delta X$

For instance, in our example, when going from $X = 2$ to $X = 3$ (so "run" $\Delta X = 1$), the Y value rises from $Y = 5$ to $Y = 8$ (so "rise" , $\Delta Y = 3$). Thus:

Slope = $\Delta Y/\Delta X = 3/1 = 3$.

Now, our example $Y = 3X + 2$ is an example of a ***linear* function** - that is, the plot is a *straight line*. That means the slope is the same everywhere along the line, it doesn't matter where we calculate it. We have seen the slope when moving from point A to point B is 3. What about the slope from point C to point D? When going from $X = 5$ to $X = 6$, (so "run" $\Delta X = 1$), the Y value rises from $Y = 17$ to $Y = 20$ (so "rise" , $\Delta Y = 3$). So the slope evaluated around here is *also* $\Delta Y/\Delta X = 3$.

In a linear function, the slope *doesn't change* with different values of X. It is the same everywhere along the line.

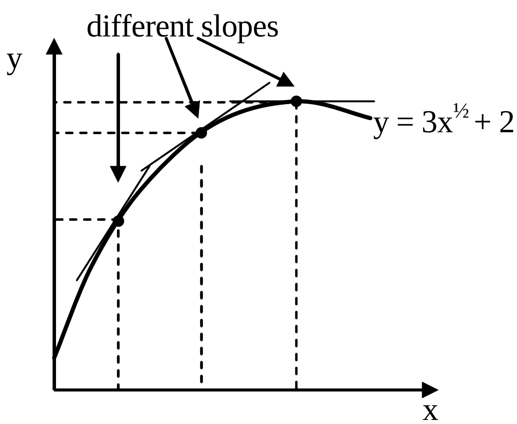
But with **non-linear functions** that is no longer true. The slope *does* change with different values of X.

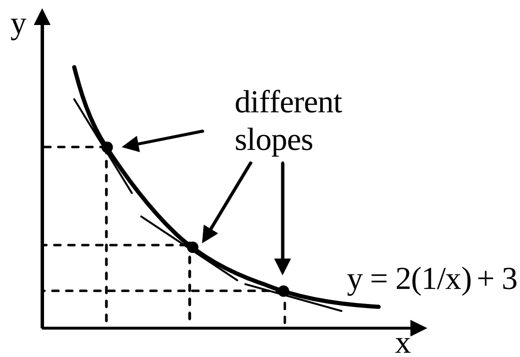
What is a non-linear function? A non-linear function is any function that does not have the simple nice linear form of $Y = bX + C$. The equation is a bit more complicated (exponents, ratios, etc).

Examples of non-linear functions:

$Y = 2X^2 + 3$ (X is squared)
$Y = 3X^3 + 12$ (X is cubed)
$Y = 9X^{0.4} + 15$ (X has the raised exponent 0.4)
$Y = 9/X + 4$ (X is in the denominator)

If you plot out a non-linear function, it doesn't come up a straight line. But rather it becomes a bent curve or squiggly line. e.g.





Non-linear functions have changing slopes - that is, the slope is different, depending on where we evaluate it. This is immediately obvious diagrammatically. For instance, in the first diagram, the slope starts off very steep then gradually gets flatter, etc.

CALCULATING SLOPES

Calculating Slopes (Tiresome Method)

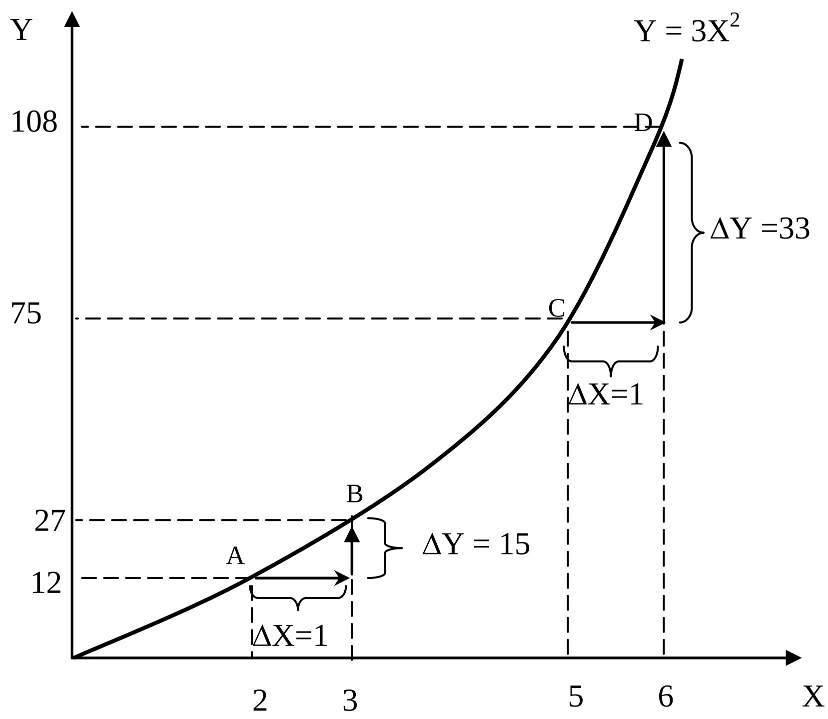
Consider the following example: . $Y = 3X^2$. Because X is 'squared' this is a non-linear function. How do we figure out the slope? Let's try to figure it out directly.

Now, $Y = 3X^2$ is still an "if...then" statement. We can still calculate the values directly (remember simple math: square first, multiply after):

if X = 0, then $Y = 3(0^2) = 3 \times 0 = 0$
if X = 1, then $Y = 3(1^2) = 3 \times 1 = 3$
if X = 2, then $Y = 3(2^2) = 3 \times 4 = 12$
if X = 3, then $Y = 3(3^2) = 3 \times 9 = 27$
if X = 4, then $Y = 3(4^2) = 3 \times 16 = 48$
if X = 5 then $Y = 3(5^2) = 3 \times 25 = 75$
if X = 6, then $Y = 3(6^2) = 3 \times 36 = 108$

and so on:

Plotted out on a diagram, it looks like the following:



It is a bent *curve* rather than a straight line. As a result, the slope is different at different levels of X.

Consider the slope around X = 2, that is going from point A to point B. As we know:

when X = 2, Y = 12
when X = 3, Y = 27.

That means so going from X = 2 to X = 3 (run = $\Delta X = 1$), Y rises from Y = 12 to Y =27 (so rise = $\Delta Y = 15$). So the slope here, "rise over run", is:

Slope evaluated around point A = $\Delta Y/\Delta X = 15/1 = 15$.

Now consider the slope further along, around X = 5, that is going from point C to point D. As we can see:

when X = 5, Y = 75
when X = 6, Y = 108.

Going from X = 5 to X = 6 (run = $\Delta X = 1$ again), the value of Y rises from Y = 75 to Y = 108 (so rise = $\Delta Y = 33$). So here:

Slope evaluated around point C = $\Delta Y/\Delta X = 33/1 = 33$.

The *slope has changed*. Evaluated around X = 2, the slope is 15 (relatively 'flat'). Evaluated around X = 5, the slope is 33 (relatively 'steep').

In non-linear functions, the slope is no longer the same everywhere. It depends *where* we evaluate it.

Calculating Slopes (Formula Method)

Happily, calculating the slope of a non-linear function isn't that tiresome. The first thing we notice is that the value of the slope *depends* on the value of X. When evaluated around X = 2, the slope was 15, when evaluated at X = 5, the slope was 33. The value of the slope varies with the value of X. Notice that that implies a sort of "if...then" relationship here too.

if X = 2, then $\Delta Y/\Delta X = 15$
if X = 5, then $\Delta Y/\Delta X = 33$.

and so on.

Well, as you know, "if..then" relationships can often be captured by a simple equation formula - in this case, linking X as the independent variable and $\Delta Y/\Delta X$ as the dependent variable. $\Delta Y/\Delta X$ is some function of X.

And that's all that ***calculus*** really is about. It is a way of figuring out what the slope formula is from the original formula.

I'll state it right here: if our original equation is $Y = 3X^2$, then let me propose (without proof) that the slope formula is:

$$\Delta Y/\Delta X = 6X$$

(I'll show you how I figured that out in a moment). Just for now, examine that formula.

First off, the slope is not constant. Notice that depending on what X is, the slope will be different. If X = 3, the slope is 18. If X = 6, then slope is 36. And so on.

Second off, it seems wrong. After all, this formula suggests that if X = 2, then $\Delta Y/\Delta X = 6(2) = 12$. But we saw from direct calculation that the slope around X = 2 was actually 15, not 12.

That is true. The difference is that my formula uses a much smaller "run". When we calculated the original slope, we assumed the "run" $\Delta X = 1$. That is, a change in X by a full unit, from X = 2 to X = 3. My formula assumes an infinitesimally small "run", that is the slope going from X = 2 to X = 2.00001. So my run is $\Delta X = 0.00001$ or something like that, a very *slight* run.

(Don't get too hung up on the meaning of infinitesimal. I am being fast and loose here).

My formula is a more "proper" way of saying "slope evaluated at around 2". That is, going from X = 2 to X = 2.000001 is really evaluated "*around 2*". Whereas going from X = 2 to X = 3, is that "around 2" or "around 3"? Going from 2 to 3 isn't really evaluating "around 2", but rather between 2 and 3. So, it is more like evaluating the slope around the midpoint between 2 and 3, that is, around 2.5. (check my formula: if X = 2.5, then $\Delta Y/\Delta X = 6(2.5) = 15$. Ta-da!)

But let's move on. We calculated the slope between X = 5 and X = 6 was 33. Well, calculate the slope around that mid-point X = 5.5, then $\Delta Y/\Delta X = 6(5.5) = 33$. Formula works!

So this simply formula $\Delta Y/\Delta X = 6X$ gives us a precise an easy way to determine what the slope is at *any* point, at *any* value of X. Just choose the X where you want to evaluate the slope, apply the formula, and we're done.

THE RULES OF CALCULUS

OK, time to get serious. *How* did I figure out that the formula for the slope was $\Delta Y/\Delta X = 6X$?

I applied the simple rules of calculus to the original formula. There are only *six* rules you need find the slope of almost every curve you'll ever come across. Apply these rules to the formula for the curve, and you'll have the formula for the slope.

(Don't wonder where these rules come from, just "have faith" that they work. If you really, really want to know where they come from, check out any calculus textbook. But it is not necessary to know *why* they work, just trust that they *do* work):

(1) **Coefficient Rule** If the original curve is $Y = bX$, where X is the independent variable and b is a multiplicative constant ('coefficient'), then the slope is $\Delta Y/\Delta X = b$.

e.g. if $Y = 6X$, then $\Delta Y/\Delta X = 6$.
e.g. if $Y = 0.5X$, then $\Delta Y/\Delta X = 0.5$.

and so on.

(2) **Constant Rule** Wherever there is an additive constant, set it to zero. Additive constants have *no* effect on slope.

So, if $Y = bX + c$, where b is the coefficient and c is an additive constant, then $\Delta Y/\Delta X = b$. The additive constant c doesn't figure in the slope calculation.

e.g. if $Y = 6X + 12$, then $\Delta Y/\Delta X = 6$. We ignore 12.
e.g. if $Y = 0.5X - 0.3$, then $\Delta Y/\Delta X = 0.5$. We ignore -0.3.

(3) **Power Rule** Wherever there is an exponent, bring it "down" as a multiplicative coefficient and subtract 1 from the exponent. So if $Y = bX^n$, then $\Delta Y/\Delta X = nbX^{n-1}$

e.g.

if $Y = 5X^3$, then $\Delta Y/\Delta X = (3 \times 5)X^{(3-1)} = 15X^2$
if $Y = 0.8X^2$, then $\Delta Y/\Delta X = (2 \times 0.8)X^{(2-1)} = (1.6)X$

Let's take a pause here. This is how I calculated the slope of our case. Our original formula was $Y = 3X^2$. There's no additive constants to worry about. But there is an exponent. So I applied the power rule:

$$\Delta Y/\Delta X = (3 \times 2)X^{(2-1)} = 6X$$

And that was all I did. Nothing more complicated than that.

Notice: the calculus applies to any function, linear or non-linear. You can immediately see that our tried-and-true graphical method of calculating the slope for a linear curve is also just an application of the rules. Consider the linear function $Y = 2X + 3$. We used it in our previous math handout and found out, by rise-over-run graphics, that the slope was 2. Well, apply the calculus rules directly to the formula. If $Y = 2X + 3$, then 2X is a multiplicative thing with no exponents, so Rule 1 applies (2X becomes simply 2) and + 3 is an additive constant so Rule 2 applies (+3 becomes 0). So if $Y = 2X + 3$, then the slope is:

$$\Delta Y/\Delta X = 2 + 0 = 2$$

i.e. the slope of our line is 2. Just as we figured out by the more tiresome graphical method. Notice there is no X in the slope formula. That means the slope of $Y = 2X + 3$ is *always* 2, no matter where it is evaluated. The slope doesn't change. It is steady everywhere. Which is also what we figured out graphically about the slopes of linear functions. The calculus works, simply and directly.

These three rules - coefficient rule, constant rule and power rule - are the three basic rules of slope calculation and all that you'll usually ever need in simple cases. But for more complicated curves, you need three more rules (which are quite uglier). For the sake of completeness, I will mention the other three rules (but you probably won't have to worry about them in practice):

(4) **<u>Sum Rule</u>**: If a particular curve is a additive combination of two different functions of X finding the slope of the total is done simply by summing the derivatives of the parts, i.e.. if $Y = f(X) + g(X)$, then $\Delta Y/\Delta X = \Delta f(X)/\Delta X + \Delta g(X)/\Delta X$.

e.g. $Y = 3X^2 + 4X^3$

then the derivative of the first part is $\Delta 3X^2/\Delta X = 6X$ (using the power rule from before) and the derivative of the second part is $\Delta 4X^3/\Delta X = 12X^2$ (again, using the power rule). So simply summing them together:

$$\Delta Y/\Delta X = 6X + 12X^2$$

and we're done. That's the slope of the curve.

If the curve *subtracts* two parts, $Y = f(X) - g(X)$, then the derivative of the whole is the *subtraction* of the derivatives of the parts.

e.g. if $Y = 5X^2 - 3X^4$

then $\Delta Y/\Delta X = 10X - 12X^3$

(5) **Product Rule** Now it gets ugly. If a particular curve is a multiplicative combination between two different functions of X then there is a complicated formula to calculate the slope.

If $Y = f(X) \times g(X)$, then:

$\Delta Y/\Delta X = [(\Delta f(X)/\Delta X) \times g(X)] + [(\Delta g(X)/\Delta X) \times f(X)]$

or, as is commonly recited by English schoolchildren, "derivative of the first times the second plus derivative of the second times the first."

e.g. suppose $Y = 7X^2 \times 4X^3$. There are two functions within the equation, the first part is $7X^2$ and the second part is $4X^3$. So what is the slope of the total function, $\Delta Y/\Delta X$?

Apply the rule. The derivative of the first part ($7X^2$) is 14X (using the earlier power rule). The derivative of the second part ($4X^3$) is $12X^2$ (again, using the power rule). Then the elements we need for our combining formula are:

derivative of the first × second = $14X \times 4X^3$
derivative of the second × first = $12X^2 \times 7X^2$

Now, if you remember how to combine exponents $(aX^n) \times (bX^m) = (a\times b)X^{(n+m)}$, then we can further simplify it:

derivative of the first × second = $14X \times 4X^3 = (14\times4)X^{1+3} = 56X^4$
derivative of the second × first = $12X^2 \times 7X^2 = (12 \times 7)X^{2+2} = 84X^4$

Finally we just use the main sum of functions rule:

$\Delta Y/\Delta X = 56X^4 + 84X^4$

which we can further simplify, by factoring out X^4, to $(56+84)X^4 = 140X^4$.

So from the complicated function:

$Y = 7X^2 \times 4X^3$

we found the slope to be merely:

$\Delta Y/\Delta X = 140X^4$

A longwinded way to get to this simple result. But not insurmountable. All you need to remember are the rules and apply them doggedly.

(6) **Quotient Rule** This is the most hideously ugly rule of all. Basically you use this rule if a particular curve is divides two different functions of X, e.g. $Y = f(X)/g(X)$. The formula to calculate the slope of this is this monstrosity:

$\Delta Y/\Delta X = \{[(\Delta f(X)/\Delta X)\times g(X)] + [(\Delta g(X)/\Delta X) \times f(X)]\}/\{[g(X)]^2\}$

Yuck.

English schoolchildren recite this as "derivative of the top times the bottom *minus* derivative of the bottom times the top, *all* divided by the square of the bottom".

If you ever do calculus more seriously, you need to recite that from memory.

e.g. Suppose $Y = 5X^2/(4X+3)$

Yuck. The top is $5X^2$ and the bottom is $4X+3$. So applying our earlier rules:

derivative of the top = 10X (product rule)
derivative of the bottom = 4 (coefficient & constant rules)

Now let's begin to apply the quotient rule:

derivative of the top × the bottom = $10X \times (4X + 3) = 40X^2 + 30X$
derivative of the bottom × the top = $4 \times 5X^2 = 20X^2$

So applying our quotient rule formula:

$\Delta Y/\Delta X = [(40X^2 + 30X) - (20X^2)]/[(4X+3)^2]$

or:

$\Delta Y/\Delta X = [20X^2 + 30X]/[(4X+3)^2]$

That's the ugly slope of the original ugly function. Hopefully you'll never come across something like this in your life again.

And that's all there's to it. You are now versed in enough calculus to handle most of any slope problems you'll ever come across.

Just six simple rules. Apply them when you need to.

More examples:

Example 1. $Y = 2X^2 + 3$

By power rule, $2X^2$ becomes 4X, by additive rule, +3 becomes 0. So:

$\Delta Y/\Delta X = 4X$

Example 2: $Y = 3X^3 + 12X$

By power rule, $3X^3$ becomes $(3 \times 3)X^{(3 - 1)} = 9X^2$, by multiplicative rule, 12X becomes 12, by summation rule:

$$\Delta Y/\Delta X = 9X^2 + 12$$

Example 3: $Y = 9X^{0.4} + 15$

By power rule, $9X^{0.4}$ becomes $(9 \times 0.4)X^{(0.4 - 1)} = (3.6)X^{-0.6}$, by additive rule + 15 becomes 0, so:

$$\Delta Y/\Delta X = (3.6)X^{-0.6}$$

Example 4: $Y = (9/X) + 4$

By quotient rule, 9/X becomes $[(1 \times X) - (1 \times 9)]/X^2$, or $(X - 9)/ X^2$, by constant rule, +4 becomes 0, so:

$$\Delta Y/\Delta X = (X - 9)/ X^2$$