ON THE DETERMINATENESS OF ECONOMIC EQUILIBRIUM

[This is a translation of an article which appeared under a slightly different title in the Giornale degli Economisti, 1891. The inquiry takes its start from a passage in the then recently published second edition of Marshall’s Principles, where he addresses from the present writer’s essay on Mathematical Psychics a construction there largely employed in the investigation of economic equilibrium, the contract-curve. Apart from this connection the discourse is not closely related to the text. For Marshall in the passage cited has in view a market in a special sense distinguished from normal, whereas the process which I analyse has much in common with the determination of normal equilibrium. Besides, as argued by Mr. Berry in the same volume (X.L.) of the Giornale, the term “determinate” is used by Marshall in a somewhat different sense from that which I have adopted.]

Apropos, it may be remarked that there is a certain indeterminateness about the use of the term “determinate” by economists. Thus Pareto has demurred to the description (above, E) of the transactions between two monopolists as indeterminate. There being more equations than unknowns, the problem, he thinks, cannot properly be described as “mathematically indeterminate.”

I dispute no man’s definition of terms; concerned, rather, with the truth of propositions. The proposition which I seek to establish here relates to a typical market consisting of two groups of individuals, say A’s and B’s: the A’s offering the commodity α in exchange for β supplied by B’s. Each A makes agreements with B’s independently of and not in concert with other A’s; and the B’s likewise not independently. The term “market” applied to this transaction is not to be understood in a sense opposed to “normal” or “natural.” Rather there is conceived to be a certain normality about the proceedings. They need not be supposed to take up a long period; rather the contrary, since the disposition and circumstances of the parties are assumed to remain throughout constant. But it is supposed
that agreements are renewed or varied many times. A "final settlement" is not reached until the market has hit upon a set of agreements which cannot be varied with advantage to all the re-contracting parties. The re-contract most favourable to the disturbance of a temporary equilibrium is one in which an A deals with a great many B's. If that power is not used, e.g., if each A confines himself to dealing with one B, it is quite possible (as will appear below) that re-contract thus hampered would not disturb the equilibrium. Thus the condition that perfect finality should be reached may be stated as follows, in the case, say, of equal numbers on both sides of the market, m A's and m B's; it must be impossible for any number of A's, say m—n, dealing (each for himself) with any number of B's to enter into a new set of agreements with advantage to all the re-contracting parties. Since in general the less restriction there is on the number of B's who re-contract the greater is the possibility of a new equilibrium, the condition is adequately expressed by the proviso that it should not be possible for (m—n) A's to re-contract with all the B's. This "all" has proved a stumbling-block to a critic who writes in the same Giornale for June, 1891. But I think it might have been difficult to recall the explanation above given by any other concise phrase. I trust that I shall not suffer like the plaintiff in the old state of the English Law who lost his case because in describing an article which had been stolen from him he spoke of a "ham" where he should have used the words "part of a ham." At any rate before condemnation is passed, reference should be made to the writer's essay on *Mathematical Psychics*; on which the article in the Giornale is largely based. In the reproduction of the article here presented I have omitted several long passages which purported to be restatements of theories more accurately enunciated in that essay.

I illustrate the theory of determinate equilibrium by two examples in which first appearances are deceptive. There is first the case in which the marginal utility of both articles for at least one of the parties varies with the terms of an agreement. This circumstance may seem at first sight unfavourable to determinateness. It may be to some kinds of determinatenesses, but not to that which has been above defined. Rather, it is the general rule that both articles should vary in respect of final utility for both parties. If we have to do not with the general problem, but with the particular one in which the marginal utility of one commodity remains constant—as in the instance cited below from Auspitz and Lieben—the equilibrium does not on that
account, I think, become more determinate than in the general case, theoretically at least, and apart from " friction."

If this view is correct, indeterminateness is not to be attributed to the labour market, because the marginal utility of money varies with the price of labour that may be set up. But a certain indeterminateness is to be attributed to that market for a quite different reason, namely, the circumstance that a man cannot, or at least does not, simultaneously serve two masters. This point is disputed by the aforesaid critic in the Giornale (June, 1891). But, as appears from the passages cited below from the Giornale, he seems not to have taken account of an essential condition in our problem, viz., that the competing work-people should not act in concert.]

The theory of Exchange is founded on the principle of Barter, which has been discussed by Marshall with remarkable originality and accuracy. He has avoided the common error of attributing to two persons who are bargaining with each other a fixed rate of exchange governing the whole transaction. A uniform rate of exchange, he remarks, is applicable only to the case of a perfect market. By way of example he puts the case of A having a basket of apples and B a basket of nuts; A desiring nuts and B apples. Referring to this example, I would express the process of barter mathematically in the following manner.

Let the abscissa \( x \) denote the number of apples given by A and received by B; the ordinates \( y \), the number of nuts given by B and received by A. Thus every point in the plane \((x, y)\) represents a barter of so many apples for so many nuts. Let \( u \) be the utility, or satisfaction, of A so far as it depends on the one hand on the number of nuts that he gains, and on the other hand the number of apples that he retains, that is the number initially in the basket less the number that he has parted with. Let \( v \) be the similarly defined advantage of B. Bartering will continue as long as it is possible for both parties to gain thereby. Let \( \Delta x \) be the quid given by A and \( \Delta y \) the pro quo received by him at any stage of the transaction. The process of exchange

\[ u = \psi(x - a) + \psi(y), \]

where \( a \) is the original stock of apples; with a corresponding expression for \( v \). But we are not limited to this simple form. We are at liberty to use for it an expression of the form \( x(a - x, y). \)
can only continue as long as the gain (of satisfaction) by A and likewise B's gain is positive; in symbols,
\[
\frac{du}{dx} \Delta x + \frac{du}{dy} \Delta y > 0, \quad \frac{dv}{dx} \Delta x + \frac{dv}{dy} \Delta y > 0.
\]
Now this condition will cease to be fulfilled when the total quantities exchanged, \(x\) and \(y\), are such as to satisfy the equation
\[
\frac{du}{dx} \frac{dv}{dy} = \frac{dv}{dx} \frac{du}{dy}.
\]
The locus thus represented I have called the contract-curve.

In Fig. 1 any point, for instance \(a\), denotes the exchange of, or contract for the exchange of, the number of nuts represented by the ordinate \(ab\) against the number of apples expressed by the abscissa \(Oa\). The series of short lines \(Oa, ab, bc, \ldots\) corresponds to successive barterers (at different rates of exchange) of a few nuts for a few apples. The broken line \(Ob'c'e'\) indicates a possible set of exchanges more favourable to B. Alike at \(e\) and at \(e'\) the bartering comes to a stop, those points being situated on the contract-curve \(PQ\). Of this curve the only part with which we are concerned is that which is intercepted by the curves of constant satisfaction, or curves of indifference, for A and B respectively which pass through \(O\); viz., \(OhP\) and \(OhQ\). The curve of indifference which passes through a given point is the locus of all the contracts that procure to the con-
tractor the same satisfaction as the contract designated by the
given point. For instance, the contract designated by the point
$h$, that is $6h$ apples given for $4h$ nuts, or, again, the contract
designated by $P$, procures for the contractor $A$ the same satis-
faction as the contract (or absence thereof) designated by $O$. It is
indifferent to $A$ whether he makes either of the first two deals,
or none at all. Similarly, it is indifferent to $B$ whether the
dealing is represented by the point $O$, or $g$, or $Q$.

At what point on the tract of contract-curve between $P$ and
$Q$ the process of bartering will come to a stop cannot be predicted.
The position of equilibrium may be described as indeterminate.
The essential condition of this indeterminateness is the absence of
competition.

The essential condition is not to be sought in an incident of
the case before us, namely, that the marginal utility of both the
commodities varies in the course of the dealing. The phenomenon
of indeterminateness may very well exist without that incident.
Whether or not the marginal utility of one commodity, say $y$, is
regarded as varying with the additions or subtractions incidental
to exchange, there will always remain—in the case of barter
between two individuals—an indeterminate tract on the contract-
curve, every point of which is a position of equilibrium. True,
the curve will sometimes degenerate into a right line parallel to
one of the axes. An example is furnished by Messrs. Auspitz
and Lieben in that part of their important book in which they
discuss the contract between a monopolist entrepreneur and a
union of operatives. Assuming that the marginal utility of
money may be treated as constant, they justly observe that
"the determination of price seems to be between wide limits
arbitrary."

Thus the imperfections of the labour market do not depend
on the circumstance that the marginal utility of money varies
for the workmen according to their bargain with the employers,
according as their wages afford only bare necessities or super-
abundant luxuries. The imperfections of that market are rather

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1 In symbols if the final utility of $y$ is constant for both dealers, we may put
for $u$ the satisfaction of the contractor $A$, the expression $\phi(x) + x\phi$, and for $v
(pertaining to $B$) $\phi(x) + \beta y$; where $\alpha$ and $\beta$ are constants. Accordingly we
obtain for the contract-curve $\forall v(x) = \forall u(x); \forall x$ constant; representing
a line parallel to the axis of $y$. This line fulfills the characteristic condition of the
contract-curve that at any point of the locus the inclination of the tangent to a
curve of indifference passing through that point should be the same for both
parties.
to be sought in certain peculiarities which Marshall has pointed out, as noticed at the conclusion of this paper.

Apropos of impediments to the play of competition in the labour market there may be observed from the point of view here adopted two incidents which are more curious than important.

Suppose that every A can contract with only one B, and likewise every B with only one A. Then it is no longer possible that \((m-n) A's\)—each acting independently—should form a set of new contracts with all—or any number of—the B's (and likewise impossible for \((m-n) B's\) to re-contract); which variation of contracts—when it can be effected with advantage to all the re-contracting parties—is here regarded as the essential attribute of competition. Accordingly, the equilibrium would be in the case supposed as indeterminate for a set of couples, as we have seen that it is in the case of a single couple. There may possibly exist types of domestic service which fulfill the supposed condition. But practical importance is not claimed for this curiosum.

There is another unobserved peculiarity of the labour-market which is the more curious in that it constitutes a positive advantage to the work-people in their dealings with entrepreneurs. Suppose that the system of contracts is initially at any point on the contract-curve (Fig. 1) \(d'\) on the right of the position hereinafter defined \(\Omega\); that is to the advantage of the B's who supply the article \(y\). It may be shown that the advantage which the B's thus possess is lost through the action of \((m-n)\) of their number who carry off, so to speak, the whole (or a large part) of the A group.* Suppose now that the A's are workpeople, the B's entrepreneurs. The terms first proposed may be very much to the advantage of the entrepreneurs. But they will lose that advantage through competition against each other. Assuming that an entrepreneur can employ several men, it will be to his advantage to offer to some of his rivals' workmen better terms than they were receiving, and so carry them off. And this process will theoretically continue until the system of agreements reaches the position of stable equilibrium symbolised by the point \(\Omega\), the point at which the demand curves (not shown in the figure) intersect on the contract-curve. Analogously it might be supposed that if the terms are at first too favourable for the workpeople, if, for instance, the point \(c\) (left of \(\Omega\)) represented an initial system of contracts, the workpeople would lose that advantage, by mutual competition. But such competition would imply, according to our analysis, that a workmen

* See note at end of article.
takes on several entrepreneurs; that a man can serve two masters simultaneously. In the case of painters, no doubt, and many home-workers plurality of employers is common. Still, the general rule is that no man can serve two masters, and so far as this is true the work-people have an advantage over the entrepreneurs in that they cannot equally beat down the price of their services by mutual competition.

I do not, however, regard these nice points as more than curious, of little practical importance in comparison with the conditions of the labour-market on which Marshall has dwelt; in particular, the tendency of any accidental disadvantage under which the work-people may be suffering to become perpetuated through the lowering of their vitality and efficiency, and the fact that employers are few in comparison with the number of work-people. In this field Marshall has thoroughly reaped the harvest, leaving nothing to those who come after him but to glean some logical niceties.

(Note referring to p. 318.)

[The argument that if a rate of exchange unduly favourable to the work-people is set up they will not beat the price down by their mutual competition is disputed by an able critic in the June number (1891) of the Giornale, on grounds of which the following quotation (loc. cit. p. 503) contains the gist. "Though it is true that 'no man can serve two masters,' yet it is quite possible for a number of work-people to increase the number of entrepreneurs whom they serve. If, on average, 100 work-people serve one employer, 500 may at pleasure renew their contract with six or with four. The first proceeding tends, according to Professor Edgeworth's reasoning, to favour the entrepreneurs, the second the work-people."

To which I reply in the October number of the Giornale (1891) suggesting that the critic has not taken account of the condition that the work-people should not set in concert.

"Say five entrepreneurs employ each on average 100 men. Four of those entrepreneurs are disposed to employ a larger number of workmen, and they offer higher wages, each finding that he can thus make a bargain more advantageous for himself and for his employees. What can be simpler? But now consider the opposite case. How can 400 work-people originally employed by four entrepreneurs find occupation with five entrepreneurs by way of an initiative on the part of the work-people, each acting independently, and not in concert with others. This case is not analogous to the first; because it is not in the power of any operative to purchase, so to speak, a fraction of an entrepreneur, other than, or in addition to, that which he already enjoys in virtue of his present engagement."]