SECTION II

THEORY OF MONOPOLY

(E)

THE PURE THEORY OF MONOPOLY

[THIS is a translation of *Teoria Pura del Monopolio* published the *Giornale degli Economisti*, 1897; itself a translation from an English original which has been lost. Much of the contents might with equal propriety have appeared in the Sections dealing with Taxation and Mathematical Economics. But it has not seemed advisable to break up the article. The theory of monopoly in the ordinary sense of the term is connected with the theory of two-sided monopoly or "duopoly." Cournot had represented the transactions between two parties to be determinate in the same sense as competitive prices. But heavy blows had been dealt on this part of his system by Bertrand in the *Journal des Savants*, 1883, and by Marshall, in an early edition of his *Principles of Economics*. Still in 1897 much of Cournot's construction remained standing; the large part which is based on the supposition that the monopolist's expenses of production obey the law of diminishing returns. Now the demolition of Cournot's theory is generally accepted. Professor Amoroso is singular in his fidelity to Cournot (*cp.* ECONOMIC JOURNAL, September 1922).]

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SECTION I.—*On the effects of a tax in the simple case of a single monopolist dealing with a group (or groups) of individuals competing against each other.*

Cournot has fully discussed the typical case in which a commodity of uniform quality is offered at one and the same price by a monopolist producer to consumers who compete against each other. The price is determined by the condition that the net gain of the monopolist should be a maximum. The quantity which is to be maximised may be represented by the expression

$$pD - \varphi(D), \text{ where } D = F(p);$$

if p is the price, $F(p)$ the quantity of the article which is demanded at the price p, and $\varphi(D)$ the cost of producing the quantity D. This formula remains applicable if we suppose that $\varphi(D)$ indicates not merely the money cost, the expenses of the monopolist, but the measure of "real cost," [1] the pecuniary equivalent of the efforts and sacrifices incurred by him in the production. Thus interpreted the formula may be extended, by simply changing the signs, to a monopolist consumer who deals with producers competing against each other. In this case $F(p)$ expresses the quantity of an article offered by competitive producers at the price p; and $\varphi(D)$ represents the total utility for the monopolist of the quantity (D).[2]

The effects on the price and on the quantity of an article which are caused by a tax are represented by the same expression in both the cases. If V is the total net utility of the monopolist, whether he is producer or consumer, then for the increment of price consequent on a small tax, for instance, u per unit of product, we have in both cases

$$uF'(p) \div \frac{d^2V}{dp^2};\ ^3$$

which expression is necessarily positive. To investigate the effect

[1] *Cp.* Marshall, *Principles of Economics*, *sub voce* "Real Cost."

[2] The total utility *simpliciter*, if the monopoly is enjoyed by an individual; but if the part of monopolist is played by a combination—for instance, a co-operative buyers' association—there should be understood the sum of the total utilities obtained by each member from the portion of commodity assigned to him; a conception which is not necessarily identical with the *Gemeinnutzen* of Auspitz and Lieben, relating to a regime of Competition.

[3] Cournot, *Recherches*, Art. 38.

of the tax on the quantity of the commodity taken at the price, it is convenient to consider the price as a function of the quantity; say $p = f(x)$.[1] * Then, if the monopolist is the producer, we have

$$V = xf(x) - \varphi(x),$$

and for the increment of x

$$\Delta x = u \div \frac{d^2V}{dx^2};$$

which expression is necessarily negative. If the monopolist is the buyer, the signs in the expression for V are changed; while the equation for Δx remains the same.

As the tax may very well be imposed not on the monopolist but on the competitive group, especially when the latter act as producers, it may be well to observe that in general it makes no difference theoretically on which of the two parties the tax is imposed.[2]

Analogous propositions may be proved for an *ad valorem* tax which is not regressive by inserting in the expression for the tax, instead of u, x as just now, any function of x (or of p) which increases (or diminishes) with the increase (or decrease) of x.[3]

I hasten to pass on to less beaten ground.

An interesting variety of the case in which the monopolist is the buyer occurs when the quantity of the commodity that is on sale is absolutely limited; for instance, when it consists of land offered by competing owners. Here $F'(p)$ is zero, and accordingly

$$dV, = d(\varphi\zeta F(p)) - pF(p), = -F(p)dp.$$

Whence, as the price is continually reduced, the net profit of the monopolist continually increases up to the point at which the sellers are beaten down to nothing—theoretically nothing, practically next to nothing.

In a case of this kind a tax on rent would not fall on the competing landlords at all, but altogether on the monopolist tenant. There occurs in this case what is erroneously supposed to occur in general, that in the phrase of Mill " the price cannot be further raised to compensate for the tax, and it must be paid from the monopoly profits." [4] †

[1] *Op. cit.* Art. 43, p. 89.

* x has been substituted for Cournot's D here and in the sequel.

[2] *Op. cit.* Art. 37.

[3] Cournot, *op. cit.* Art. 41. Marshall, *Principles*, 3rd ed., p. 433, note.

[4] Mill, *Political Economy*, Book V. ch. iv. § 6. He is followed by some eminent writers, but naturally not by any of the mathematical school. See Cournot, *Recherches*, ch. vi., and Marshall, *Principles*, Book V. ch. xiii. ed. 3.

† It may be recalled, however, that, though the monopolist has an interest

It may now be asked: Will the case be materially altered if, between the monopolist buyer and the group that is under the necessity of selling without a reserved price, there is interposed a third party, namely, another competitive group with an ordinary degree of "elasticity." Where there are two groups each consisting of individuals competing against each other the introduction of a third group completely changes the incidence of a tax. Thus a tax on the ground rent of cultivable land will in general fall entirely on the owner; a tax on agricultural produce will not in general fall entirely on the owner. Does there exist a similar distinction in the case of monopoly?

It will be well to begin with the case in which all the individuals in each group are competitors; as the classical writers have hardly discussed this case in all its generality, having limited it by the special supposition that the commodity of which the supply is fixed is not all of the same quality.[1] Let us start with the supposition of three islands, A, B, C, which carry on an international trade of the following description. A buys from B goods, say *b*, for the production of which B must buy from C certain materials or "agents of production," say *c*; which are periodically supplied to C in constant quantities not capable of being increased by human effort—for instance, seaweed deposited on the shores of C. Let p_1 be the price of *b*, and p_2 that of *c*. Considering any particular producer in B, let us denote by z_t the quantity of finished goods offered by him to inhabitants of A; and by ζ_t the quantity of raw material or agent of production demanded by him from inhabitants of C. Then the net advantage of this individual, say u_t, increases with the net profit $z_tp_1 - \zeta_tp_2$; *ceteris paribus*, and abstraction being made of the efforts and sacrifices involved in the increase of production. Likewise the advantage diminishes with the increase of z_t and increases with the increase of ζ_t (the increase of material facilitating production) in virtue of these efforts and sacrifices; abstraction being made of the satisfaction resulting from increased gain. These relations may be thus expressed:—

$$u_t = F_t(+(z_tp_1 - \zeta_tp_2), -z_t, +\zeta_t).$$

As z_t and ζ_t are both controlled by the individual, he will vary

in reducing the tax, it is not a very great interest, for a reason pointed out below. [*Cp.* ECONOMIC JOURNAL, 1922, p. 439.]

[1] Ricardo in his discussion of taxes on raw material introduces at the beginning a phrase applicable to the general case, "that capital which pays no rent" (*Political Economy*, ch. ix. par. 1). But he immediately proceeds to suppose land of different qualities. Cp. Mill, *Political Economy*, Book V. ch. iv. § 3.

them up to the point at which u_t is a maximum. We have thus the two equations :—

$$(a)\ \frac{du_t}{dz_t} = 0 \quad (b)\ \frac{du_t}{d\zeta_t} = 0.$$

Eliminating ζ_t from the equations (a) and (b), we might obtain an equation of the form $z_t = \varphi_t(p_1, p_2)$, representing the offer of b by the individual No. t in B (at the prices p_1 and p_2). Summing the offers of all the producers in B, we have

$$Sz = S\varphi(p_1, p_2) = \text{say } \Phi(p_1, p_2).$$

This offer ought to be equal to the demand in A for b at the price p_1; say (1) $\Phi(p_1, p_2) = F(p_1)$. Again, eliminating z_t from the equations (a) and (b) we might obtain an equation of the form $\zeta_t = \psi(p_1, p_2)$. Whence as the sum of the ζ's is constant we have an equation of the form

(2) $\Psi(p_1\ p_2) = K$; where K is a constant.

To investigate the effect of a tax, say of u per unit of seaweed, or use of land or other limited commodity obtained from C, it is proper to put $(p_2 + u)$ for p_2 in the equations (1) and (2) from which the two prices are determined. It is evident that the value of p_1 which is obtained by eliminating the other variable is not altered by the change; the tax falls entirely on the inhabitants of C.

To study the effect of a like tax on the produce of B we ought to substitute for p_1, $(p_1 + u)$ in the left-hand member of the equation (1), or $(p_1 - u)$ in the right-hand member. In general the offer of b expressed by Φ will fall; and consequently the demand for c expressed by Ψ. The quantity of c being fixed, the fall on the demand for it is attended with a fall in its price. There is a limiting case in which the price of c is not altered, and the entire tax falls upon A. This occurs when the demand on the part of A for b is perfectly inelastic. Then $F(p_1 + u) = F(p_1)$. And so the price paid to the producers in B and their demand for c remain unaltered. Everything goes on as before, except that the inhabitants of A pay the tax in addition to the price of b.[1]

We have now to consider how these relations are modified when it is supposed that b is bought by a monopolist. Equation (2) remains as before; but for equation (1) we ought to

[1] *Cp.* II. 134.

substitute the condition that V, the net advantage of the monopolist, should be a maximum; and V is of the form

$$\Theta[\Phi(p_1, p_2)] - p_1\Phi(p_1, p_2).$$

Whence it follows that we ought to equate to zero the differential coefficient of the expression $V - \lambda(\Psi - K)$ (where λ is the undetermined multiplier proper to problems of relative maximum) —the complete differential, not simply the partial differential with respect to the variable which is directly under the control of the monopolist, viz. p_1. For why should the monopolist stop at the value of p_1 which is given by the equation $\left(\frac{dV}{dp_1}\right) = 0$; p_2 not varying. He will go on making p_1 to vary directly and p_2 in virtue of equation (2), indirectly up to the point at which V cannot be increased by any variation of p_1 consistent with equation (2).

It appears from this analysis that, as before, a tax on c will fall entirely on C. With respect to a tax on b, the case of monopoly agrees with that of competition in this respect, that in general the price of c will be somewhat reduced.

Suppose now that either of the groups B or C becomes solidified as a monopolist. Presumably each monopolist will fix the price which is directly under his control at that figure which he thinks likely to afford him the greatest net advantage, account being had of the price which will probably be fixed by the other monopolist for the article under his control. It is thus that the stroke of a fencer is influenced by his prevision of what his adversary's parry will be. The economic fencing-match may continue till one of the fencers is ruined. Pure theory does not seem to assign any stage at which they must stop.

This is a particular case of the general proposition that, when more than one monopolist takes part in a system of bargains, value is indeterminate. The proof of this proposition presents a difficulty which must be overcome before we can proceed to the more complicated cases of value in a regime of monopoly.

SECTION II.—*Proof of the proposition that when two or more monopolists are dealing with competitive groups, economic equilibrium is indeterminate.*

To establish this proposition it will suffice to consider the typical cases formed by two monopolists, each of whom, acting independently, offers to a competitive group one of two articles that are either (A) *rival* or (B) *complementary* as objects of demand.[1]

A. The simplest case under this head is that in which the rival articles are not merely substitutes for each other, but actually identical. This case is treated by Cournot [2] as the first step in the transition from monopoly to perfect competition. He concludes that a determinate proposition of equilibrium defined by certain quantities of the articles will be reached. Cournot's conclusion has been shown to be erroneous by Bertrand [3] for the

[1] I define these terms as follows. I assume, notwithstanding the objections raised by some distinguished economists, in particular Prof. V. Pareto in the *Giornale degli Economisti* [*cp. Manuel*], and Prof. Irving Fisher in his *Mathematical Investigations*, p. 89, that for every system of quantities assigned to the two articles, that is, for every pair of x and y (at any rate for values above a certain minimum of these commodities—*cp.* Marshall, *Principles*, Appendix, Note vi, and passage there referred to), there is for each individual a money measure of the total utility which he derives from the consumption of assigned quantities (x and y), a measure represented by a function of those quantities (see Dupuit, article "Utility," *Journal des Economistes*, 1853).

If x and y are the quantities sold at the prices ξ and η, we have $\xi = \frac{dF_r}{dx_r}$, $\eta = \frac{dF_r}{dy_r}$ for each individual; x_r and y_r being the quantities purchased by the individual numbered r. Whence, if $F(x, y)$ is put for $\Sigma\, F_r(x_r, y_r)$—corresponding to the "Gesammtnützlichkeit" of Messrs. Auspitz and Lieben—$\xi = \frac{dF}{dx}$, $\eta = \frac{dF}{dy}$.

Well then, the articles are rival or complementary objects of demand according as $\frac{d^2F}{dxdy}$ is negative or positive. We shall have the first case when $\frac{d^2F_r}{dx_r dy_r}$ is negative for every individual (or at least on an average); the second case when that expression is positive.

From the last two paragraphs we deduce that $\frac{d\xi}{dy} = \frac{d\eta}{dx}$ is negative for rival and positive for complementary articles. Also, if x and y are considered as functions of ξ and η which may be obtained from the above given values of ξ and η in terms of x and y, it will be found that $\frac{dx}{d\eta}$ and $\frac{dy}{d\xi}$ are positive for rival and negative for complementary articles. The proof of this proposition involves the condition

$$\left(\frac{d\xi}{dx}\right)\left(\frac{d\eta}{dy}\right) - \left(\frac{d\xi}{dy}\right)\left(\frac{d\eta}{dx}\right) > 0.$$

This condition follows from the condition that in equilibrium the total utility of each individual ought to be a maximum because otherwise he will continue to buy at the prices ξ and η. Whence it is deducible that the total utility $F(xy)$ ought to be a maximum. Whence

$$\left(\frac{d^2F}{dx^2}\right)\left(\frac{d^2F}{dy^2}\right) - \left(\frac{d^2F}{dx,\, dy}\right)^2 > 0,$$

which is identical with the said condition (*cp.* below, Sect. III.).

[2] *Op. cit.* ch. vii.

[3] *Journal des Savants*, 1883.

case in which there is no cost of production; by Professor Marshall [1] for the case in which the cost follows the law of increasing returns; and by the present writer [2] for the case in which the cost follows the law of diminishing returns.

In the last case there will be an indeterminate tract through which the index of value will oscillate, or rather will vibrate irregularly for an indefinite length of time. There will never be reached that determinate position of equilibrium which is characteristic of perfect competition defined by the condition that no individual in any group, whether of buyers or sellers, can make a new contract with individuals in other groups, such that all the re-contracting parties should be better off than they were under the preceding system of contracts.

The theory may be illustrated by the extreme case of decreasing returns,* the case in which there is a fixed limit to the amount that can be produced. Suppose, for instance, that there are two monopolists, each owning a spring of mineral water (Cournot's "source minérale"), the output of which per day is limited to a certain quantity, the same for both springs. To further simplify the example, suppose that the delivery of the commodity is not attended with any expense. Further, let the demand-curve be the same for every consumer; and that the simplest possible, namely, a right line. Thus let $x_r = 1 - p$ where p is the price and x_r is the amount of the commodity demanded at that price by any individual. Accordingly, if x is the collective demand of a set of customers numbering n,

$$x = n(1 - p).$$

In Fig. 1 let us represent x by a horizontal abscissa, and p by a vertical ordinate, in accordance with Marshall's well-known construction. We may begin with the supposition that each monopolist deals with only half the total number of potential customers; which is, say, $2N$. The collective demand-curve for one of the

[1] *Principles of Economics*, first ed., note to p. 485.

[2] *Mathematical Psychics.* The competition of the two monopolists will reduce the price below the point Q in the figure on p. 114 (*op. cit.*) to within the tract between Q and T. *Cp.* note to p. 116, where the statement that "the system will reach a final settlement at some intermediate point" is inaccurate. Suppose that there are two B's dealing with an indefinite number of A's, as in the case now under consideration. The B's will force each other below the point Q; and between that point and T the position of (temporary) equilibrium will continue to vary; since it will always be the interest of one or more of the A's to re-contract with one or both of the B's; getting on to the partial or "supplementary contract curves" which are indicated at p. 37 (*op. cit.*), but not represented in the figure on p. 114.

* For an example not thus limited see ECONOMIC JOURNAL, September 1922.

markets thus constituted may be represented by the right line RC, making an angle of forty-five degrees with each of the co-ordinates, if the units pertaining to the co-ordinates are properly taken. Let OR be the unit of price. Let OA represent the quantity of commodity which is demanded at the price OP $(= \frac{1}{2})$, $= \frac{1}{2}c$ (units of commodity). Let OB, $= \frac{3}{4}c$, be the amount to which the daily output of the monopolist is limited. Let RC' likewise represent the demand-curve of the N customers who are supposed initially to be dealing with the other monopolist; with similar conventions as to the abscissa OA', OB', OC' $(= OA, OB, OC$ respectively).

Now if each monopolist were dealing independently of the other with half of the customers he would fix the price at OP, since his net profit $Np(1 - p)$ is a maximum. When $p = \frac{1}{2}$ the

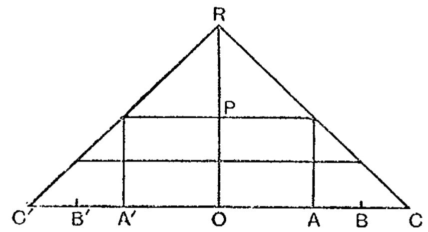

FIG. 1.

corresponding quantity would be $\frac{1}{2}c$. Let us start from this position. If the commodities were quite uncorrelated we would stop there. But as things are, it will be the interest of one of the monopolists to lower his price by a little, say δp, so as to attract his rival's customers. Throwing his whole stock on the market, he would realise a greater profit than before, namely, $\frac{3}{4}c(\frac{1}{2} - \delta p)$. He would not indeed be able with his limited supply to satisfy the entire demand, namely $c(\frac{1}{2} - \delta p)$, evoked by the lowered price. But he would have deprived his rival of a great part of his initial custom. However, the rival will now follow suit with a still lower price. So by successive steps, by variations of price which may be supposed to occur from day to day, the price may be lowered to OQ, $= \frac{1}{4}$, which is just sufficient to take off the whole supply of one monopolist offered to half the market, consisting of N customers. At this point it might seem that equilibrium would have been reached. Certainly it is not the interest of either monopolist to lower the price still further.

But it is the interest of each to raise it. At the price $\frac{1}{4}$ set by one of the monopolists he is able to serve only N customers (say the first N on a queue) out of the total number $2N$. The remaining N will be glad to be served at any price (short of unity, $= OR$). The other monopolist may therefore serve this remainder at the price most advantageous to himself, namely $\frac{1}{2}$. He need not fear the competition of his rival, since that rival has already done his worst by putting his whole supply on the market. The best that the rival can now do in his own interest is to follow the example set him and raise his price to $\frac{1}{2}$. And so we return to the position from which we started and are ready to begin a new

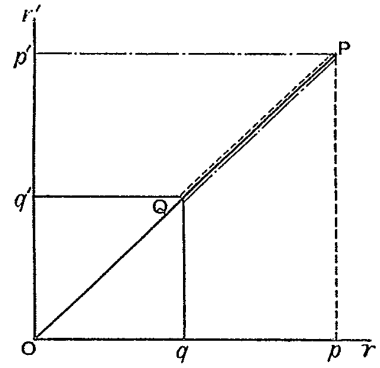

FIG. 2.

cycle. This need not have exactly the same path as that which we have described. For at every stage in the fall of price, and before it has reached its limiting value $\frac{1}{4}$, it is competent to each monopolist to deliberate whether it will pay him better to lower the price against his rival as already described, or rather to raise it to a higher, perhaps the initial, level for that remainder of customers of which he cannot be deprived by his rival (owing to the latter's limitation of supply). Long before the lowest point has been reached, that alternative will have become more advantageous than the course first described.

The matter may be put in a clearer light by taking ξ and η as co-ordinates representing the prices of the articles which are in the limiting case now considered identical, but in general only rival. The *dotted* lines in Fig. 2 represent the locus of maximum profit for the monopolist owning the commodity x, of which the price is ξ—the *watershed*, so to speak, of the utility surface for that monopolist (or more exactly the locus of that price of x which for any assigned price of η affords maximum profit to the owner of x).* The corresponding locus for the second monopolist is represented by the *broken* lines. Corresponding to the data above defined, put $Op = Op' = \frac{1}{2}$; $Oq = Oq' = \frac{1}{4}$.

If we start from a price Or, above Op, the same for both, it will be the interest of one monopolist to lower his price to Op. The other monopolist from a similar motive, and faced with the loss of custom, follows suit. So we come to the point P, the position of equilibrium if the two markets were separate, or if the two monopolists were in combination. Now it is the interest of the seller of x to lower his price by a little and so (the price of y remaining the same) to move to the point where the dotted line parallel to PQ intersects the broken line Pp'. The seller of y then lowers his price η to a point on the broken line which hugs the diagonal on the right. And so the system may dance down to the point which corresponds to the price Oq ($= Oq'$), below which there is no tendency for the price to be lowered. But before this limit has been reached, the first price may have jumped back to the border-line Pp. The second will then presumably jump on to the line Pp'; and so *perpetual motion* is set up.

It will readily be understood that the extent of indeterminateness diminishes with the diminution of the degree of correlation between the articles. The illustration above given may be adapted to exhibit this incident.†

In the limiting case of no correlation between the commodities the locus of maximum advantage for each monopolist becomes a line parallel to one of the axes. For instance, if Oa in Fig. 3 is the value of ξ which affords maximum profit to the owner of x when η, the price of y, is zero, Oa continues to do so when η varies; aQ is the locus of maximum advantage for the owner of x.

B. The case of complementary demand may be illustrated by

* In general the maximum value of ξ would depend not only on the assigned value of η, but also on the value of y.

† The figure is adapted only to cases which are adjacent to the one discussed in the text: suppose two sources of just distinguishable mineral waters which are supplied by two competing monopolists without cost of production. Some notion of the complications which arise when these simplifying suppositions are removed may be obtained from the example considered in the ECONOMIC JOURNAL, 1922; where it should be remarked that the quantities supplied, not as here the prices, are taken as the variables.

supposing $2N$ homogeneous customers whose laws of demand are for the first article :—

$$x = 2N(1 - \xi - \alpha\eta),$$

and for the second article :—

$$y = 2N(1 - \beta\xi - \eta).$$

To begin with, α and β may be supposed very small and equal. Then the loci of maximum profit are for the respective monopolists :—

$$1 - 2\xi - \alpha\eta = 0;$$
$$1 - \alpha\xi - 2\eta = 0.$$

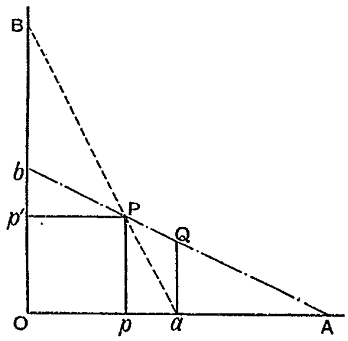

FIG. 3.

They may be imagined (they are not shown) as passing respectively through A and a in Fig. 3; the first almost vertical, the second almost horizontal.

The limiting case in which the two articles are perfectly complementary may be represented by putting α equal to 1. This is the case considered by Cournot when he supposes that each commodity has only one use; namely, to enter in a fixed proportion into the composition of a certain article for which there is a demand.[1] There are then given two monopolists who

[1] *Recherches*, ch. ix. It may excite surprise that when Cournot treats of two monopolists dealing in two perfectly rival articles, he supposes the steps towards equilibrium to be made by varying one *quantity* while the other remains constant (ch. vii.); whereas when he treats of two monopolists dealing in two articles perfectly complementary, he supposes that the steps are made by varying one of the *prices* while the other remains constant. An explanation may be found in the term "perfectly." If the articles are perfectly rival (that is, identical) there cannot well be supposed two prices; and if the articles are perfectly complementary (as in the case to which this note refers) there cannot be supposed two (independent variations of the) quantities.

offer in a market of competitive purchasers two complementary articles which enter in definite proportions $m_1 : m_2$ into the composition of an article for which the demand is $F(p)$, or $F(m_1p_1 + m_2p_2)$, where p_1 and p_2 are the prices of the complementary articles.* Abstracting cost of production we have for U the gain of one monopolist, and V the gain of the other :—

$$U = p_1F(m_1p_1 + m_2p_2);$$
$$V = p_2F(m_1p_1 + m_2p_2).$$

According to Cournot the prices are determined by the simultaneous equations

$$(1)\ \left(\frac{dU}{dp_1}\right) = 0 \quad (2)\ \left(\frac{dV}{dp_2}\right) = 0$$

(the price p_1 only being varied in U, and p_2 only in V). To which it may be objected that these equations cannot hold good simultaneously. Suppose, for instance, that the first holds good, then the second will not. For why should the second monopolist stop at the point at which the partial differential coefficient $\left(\frac{dV}{dp_2}\right) = 0$? He will go on varying the price p_2 up to the point at which the *complete* differential coefficient of V is zero. That is

$$\left(\frac{dV}{dp_2}\right) + \left(\frac{dV}{dp_1}\right)\frac{dp_1}{dp_2} = 0;$$

where $\frac{dp_1}{dp_2}$ is derived from equation (1). This equation combined with (1) will determine p_1 and p_2.

To adapt Cournot's illustration to our scheme of rectilinear demand-curves, we may, without loss of generality put

$$m_1 = m_2 = 1;$$

and write for the demand of the first commodity

$$x = 2N(1 - \xi - \eta);$$

and likewise for the second commodity

$$\eta = 2N(1 - \xi - \eta);$$

where ξ and η are the respective prices. Then the position of maximum profit to the seller of x for any assigned value of η is given by the equation

$$1 - 2\xi - \eta = 0,$$

* The reader may like to have a reference to a real case of complementary articles (links in a chain of canals) owned by different (monopolist) companies; of which one fixes a high rate which " obliges the other companies to reduce their rates." Report on Railways and Canals Amalgamation, 1846, p. 200 (Vol. XIII.), Part IV. The concrete case is, however, not so simple as the one above imagined.

represented by the dotted line *Ba* in the figure; and the corresponding locus for the seller of y is

$$1 - \xi - 2\eta = 0.$$

If we suppose that both these equations exist simultaneously we ought to have $\xi = \eta = Op = Op'$; with P as the position of equilibrium. If we suppose that one only of the equations holds good, the second, for example, but not the other, then, the first monopolist varying his price consistently with the satisfaction of the second equation, the position of equilibrium will be such that $\xi(1 - \xi - \eta)$ should be a *maximum, subject to the condition* that $1 - \xi - 2y = 0$; which gives that point on the line *Ab* for which $\xi = \frac{1}{2}$; that is, the point Q, if $OA = 1$. But it is the better opinion, I think, that neither of these suppositions is tenable. For clearly in the case of a single monopolist, when it is laid down as a fundamental principle that $pF(p)$ (less cost of production) should be a maximum, it is not supposed that the demand $F(p)$ should be subject to the condition that the prices of all the other articles should remain constant when there are other articles whose prices vary with p. We have already had an example in the international trade above described.* Here is a further somewhat fanciful illustration.

Suppose Nansen and Johansen are dragging their sledge over the Arctic plains (all their dogs having died). In the pursuit of different scientific aims one of them, Nansen, tries to get up on the ice as far as possible above the level of the sea, while the other strives to reach the position at which the depth of the sea measured from the sea-level is a maximum. With these different objects Nansen and Johansen do not act in concert; so much only of their old partnership remains that they do not act against each other, Nansen moves only in a line of latitude (in either direction), Johansen only in a line perpendicular thereto, a line of longitude, parallel to axis OB.

Under these conditions it is very possible that the two surfaces —of the ice and of the bottom of the sea—are crumpled in such wise that the sledge will never come to a point such that neither of the parties will want to get away from it. Such was the case above described with reference to rival commodities.

There is also possible another case. Suppose the principal ridge of ice on which Nansen wants to get as high as possible runs in the direction *Ba* (Fig. 3), and that the principal valley in the bottom of the sea above which Johansen wants to get as

* See above, p. 114.

high as possible runs in the direction bA. The intersection of these two lines at the point P might seem to be a position of equilibrium. From this point it is not the interest of Nansen to move either to the right or left; nor of Johansen to move either up or down. Nevertheless if Nansen were to move from this position—whether by accident in the polar darkness, or designedly foreseeing a future move of Johansen—say to the right to a neighbouring point P_1, not shown in the figure; then Johansen would tend to move downwards on the vertical line through P_1 to a point Q_2, where the depth of the valley measured from the sea-level is greatest. At this point it is probable that the height of the ice is greater than it was at the initial point P, since the "hog's-back" formed by the ice becomes higher as one moves towards OA along its crest, and accordingly as one moves near the crest, in that downward direction. Nansen will then be in a position to repeat his step to the right, whether induced by a knowledge of Johansen's motives, or simply by the fact that his first step to the right resulted in advantage to himself.* And so there may be reached a point on the line bA considerably below and to the right of the initial point P, the point at which it will no longer prove to the advantage of Nansen to take a step to the right. At this point, which proves to be that at which $\xi = \frac{1}{2}$, the point Q in Fig. 3, it may be thought that equilibrium will finally have been reached.

But it will not be a stable equilibrium, except on the extreme supposition that Nansen is perfectly intelligent and foreseeing, while Johansen, as the saying is, "cannot see beyond his nose." Otherwise let us suppose *first* that both proceed by tentative steps in the dark. At the point Q, or perhaps before getting so far from P, the immediate interest of Nansen may prompt him

* Let Q_1 be the point on the line Ba at which Johansen moving downwards from the point P_1 stops. The step PP_1 being short the position Q_1 must be more advantageous for Nansen than P from which he started. For Q_1 is *within* the curve of constant advantage, the indifference-curve, may we say, pertaining to Nansen defined by the equation $U =$ constant where $U = \xi 2N(1 - \xi - \eta)$. Whence

$$\frac{d\eta}{d\xi} = -\frac{du}{d\xi}\Big/\frac{du}{d\eta} = -\frac{1 - 2\xi - \eta}{-\xi}.$$

Thus the tangent to Nansen's indifference-curve (which is concave towards the axis $O\alpha$) is horizontal at P (since at that point $\xi = \eta = \frac{1}{3}$); and at P_1 it slopes slightly downwards to the right, but not nearly so much as the line bA. Accordingly, if Nansen makes a second short step to the right from Q_1, say to P_2, and thence Johansen moves down to Q_2 on bA, the position Q_2 will be more advantageous for Nansen than Q_1. And it can be shown that this downward movement may continue on to the point Q where the tangent to the indifference-curve becomes coincident with the line bA.

to move to the left to a point on the line *Ba*. From this point Johansen will move upwards to a point on the line *bA*. And so on, until they regain perhaps the initial position *P*; ready to start on a second excursion, this time perhaps in an upward direction. *Secondly*, let us suppose that both are perfectly intelligent and aware of each other's motives. Then from the point *Q*, for instance, it is quite possible that Johansen may move *upwards;* not that it is his immediate interest to move in this direction, but in the hope of inducing Nansen to move to the left to a position (on the line *bA*) more advantageous to Johansen than *Q*. Nansen, however, may not lend himself to this plan. And so the two may continue to make moves against each other; or if they stop, it will be only for a time, and not in a determinate position.*

To drop metaphor, it is certain in the case of rival articles offered by monopolists not in combination, and at least very probable in the case of complementary articles, that economic equilibrium is indeterminate.

It is unnecessary to point out how prevalent in the actual world are the relations of "rival" and "complementary." Let the reader consider the passages referred to in Professor Marshall's *Principles* under headings "Joint Demand," and "Substitutes." It will be sufficient here to mention two cases which, though they do not possess the essential characteristics of rival or complementary goods as above defined, yet have the property of rendering monopoly price unstable. The *summa genera* of necessary articles, food, clothes and so forth, may be regarded as complementary in a certain sense in so far as an increase in the price of one class tends to diminish that of the other class. For instance, it is said that during a dearth in one of our northern cities the price of old clothes diminished. Articles of consumption may also be rivals in a sense, though not capable of acting as substitutes for each other, if an increase in the price of one causes less money to be spent on it, and the money thus set free goes to increase the price of other articles.†

SECTION III.—Since then there is no theory of economic equilibrium in the case with which we have to do with different monopolists, we may confine ourselves to the case in which there is only one monopolist in the field. An important variety of this case occurs when there are two or three different markets

* For further illustrations of the indeterminateness which is characteristic duopoly see ECONOMIC JOURNAL, September 1922.

† *Cp.* below, p. 137.

furnished by one and the same monopolist with two or more articles of which the production is *joint* in this sense, that the increase of one renders the increase of the other (*a*) more, or (*b*) less, costly. In symbols let x and y be the respective quantities produced and $\varphi(x,y)$ the expenses, or, more generally, the pecuniary measure of the real cost of the productions of x and y together; we have then case (*a*) if $\frac{d^2\varphi}{dx,dy}$ is positive; if it is negative, case (*b*). These relations may be designated by the terms (*a*) rival production, (*b*) complementary production.

It should be observed that "complementary production" as here defined is not identical with joint production as used by some distinguished writers. If the expense incident to the production of two articles in the quantities x and y is $C + ax + by$, where C, a and b are constants, these articles would commonly be described as produced jointly, but they are not "complementary in our sense.*

(*a*) First, the production being rival, let the cost of producing x together with y be $\varphi(x,y)$ where $\frac{d^2\varphi}{dx^2}$, $\frac{d^2\varphi}{dy^2}$ (the law of increasing cost being assumed) and $\frac{d^2\varphi}{dx\,dy}$ are each positive. Let $f_1(x)$ be the price at which the quantity x is demanded in one market and $f_2(y)$ the price at which the quantity y is demanded in another market; then if V is the net advantage of the monopolist,

$$V = xf_1(x) + yf_2(y) - \varphi(x,y).$$

Now suppose a small tax of u per unit is imposed on the first commodity. If dx and dy are the consequent variations in the quantities furnished we have,

since $$\left(\frac{dV}{dx}\right) = 0, \text{ and } \left(\frac{dV}{dy}\right) = 0,$$

$$dx\frac{d^2V}{dx^2} + dy\frac{d^2V}{dx\,dy} = u,$$

$$dx\frac{d^2V}{dx\,dy} + dy\frac{d^2V}{dy^2} = 0;$$

whence $dx = u\frac{d^2V}{dy^2} \div \Delta$; $dy = -u\frac{d^2V}{dx\,dy} \div \Delta$; where Δ is the determinant $\frac{d^2V}{dx^2}\cdot\frac{d^2V}{dy^2} - \left(\frac{d^2V}{dx\,dy}\right)^2$, a quantity which must be positive in order that V should be a maximum. For the same

* Compare the definitions adopted by Pigou; as to which see passage referred to in Index, s.v. *Joint Production*.

reason $\frac{d^2V}{dx^2}$ must be negative. Also $\frac{d^2V}{dx,dy} = -\frac{d^2\varphi}{dx,dy}$ must be negative. Accordingly dx is negative, dy is positive; the purchasers of the taxed article are damnified, while the purchasers of the tax-free article are benefited. The proposition may be extended to any number of articles.

We might have reached the same conclusion if we had treated the *prices*, say ξ and η, as the independent variables; in which case it would be proper to substitute for x, $F_1(\xi)$ and for y, $F_2(\eta)$, likewise for $f_1(x)$ and $f_2(y)$ respectively ξ and η.

Analytical geometry may be usefully employed with either set of variables. Thus let V be represented by the height of a surface depending on the independent variables ξ and η. The position of maximum height is given by the simultaneous equations

$$(1)\ \left(\frac{dV}{d\xi}\right) = 0; \quad (2)\ \left(\frac{dV}{d\eta}\right) = 0$$

These equations are adequately represented, with respect to values of the variables, in the neighbourhood of the maximum by the curves AA' and BB' in Fig. 4. For both the curves in the neighbourhood of P will be inclined negatively to the axis of ξ; that is, the tangent $\frac{d\eta}{d\xi}$ will be for both negative. Further, that tangent *in absolute quantity* will be greater for AA' than for BB'. For with respect to AA'

$$\frac{d\eta}{d\xi} = -\left(\frac{d^2V}{d\xi^2}\right) \div \left(\frac{d^2V}{d\xi\, d\eta}\right)\left(\text{since } \left(\frac{dV}{d\xi}\right) = 0\right)$$

The numerator of this fraction is positive since V is a maximum (at the point P). Also the denominator is negative, as may be seen by substituting $F_1(\xi)$ and $F_2(\eta)$ for x and y in $\varphi(x,y)$. By parity of reasoning the tangent for BB'

$$= -\left(\frac{d^2V}{d\xi\, d\eta}\right) \div \left(\frac{d^2V}{d\eta^2}\right) < 0.$$

These values of $\left(\frac{d\eta}{d\xi}\right)_1$ and $\left(\frac{d\eta}{d\xi}\right)_2$, as they may respectively be called, are now to be combined with the *third* condition required in order that V may be a maximum, viz.

$\frac{d^2V}{d\xi^2} \cdot \frac{d^2V}{d\eta^2} > \left(\frac{d^2V}{d\xi\, d\eta}\right)^2$. Whence $-\left(\frac{d\eta}{d\xi}\right)_1 \Big/ -\left(\frac{d\eta}{d\xi}\right)_2 > 1$; and $\left[\left(\frac{d\eta}{d\xi}\right)\right]_1$ in absolute quantity, $> \left[\left(\frac{d\eta}{d\xi}\right)\right]_2$. Thus $\left(\frac{d\eta}{d\xi}\right)_1$ and $\left(\frac{d\eta}{d\xi}\right)_2$

being both negative, the curves ought to be (in the neighbourhood of the *maximum*) inclined to the axes and to each other as represented in Fig. 4.

Now let a (small) tax of u *ad valorem** be imposed on the x commodity. The curve AA' will be displaced to the right as in the figure; while the curve BB' remains unchanged. Thus while ξ is increased η is diminished; a conclusion identical with that reached above, since the prices and quantities vary inversely.*

If we had treated x and y as the independent variables, the loci $\left(\frac{dV}{dx}\right) = 0$, and $\left(\frac{dV}{dy}\right) = 0$ would still have been related like

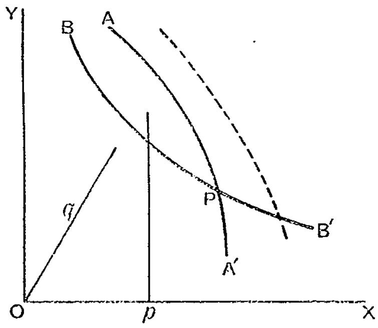

FIG. 4.

the curves AA' and BB' in Fig. 4; while the displaced curve would lie on the *left* of AA'.

Conversely it may be shown that a bounty to one of the commodities will prejudice the consumers of the other.

The effects of other kinds of governmental control may be studied by a similar procedure. Thus let there be prescribed a *maximum* price or a *fixed* price for one of the articles, a price less than what would have been reached if monopoly were allowed free play. If ξ in Fig. 4 is limited to Op, less than OP, the position of equilibrium will be the highest point on the curve formed by the intersection of the surface ($z = V$) with a plane through p perpendicular to the axis OX. The ordinate of the curve BB' formed by its intersection with a perpendicular through p (in the plane of $\xi\eta$)

* The conclusion is readily extended to a specific tax by substituting in the above for $\Delta\xi$, considered as a small percentage of ξ, the increment $\Delta x \frac{d\xi}{dx}$. Like reasoning applies to other small taxes.

to the axis OX (the price η) will evidently be greater than the ordinate intersecting at P. Of course if Op is considerably less than OP it might happen that the vertical plane through p does not meet the surface above the plane of $\xi\eta$. The value of V then becomes negative or impossible; the business cannot go on.

Very similar is the effect of the condition that one price should not exceed the other by more than a certain proportion. This condition is exemplified by the American short-haul clause; which enacts that if D_1 the distance of one station (from the terminus) is less than D_2, the distance of another station, and $\xi\eta$ are the respective fares per mile, then $D_1\xi$ shall not exceed $D_2\eta$.

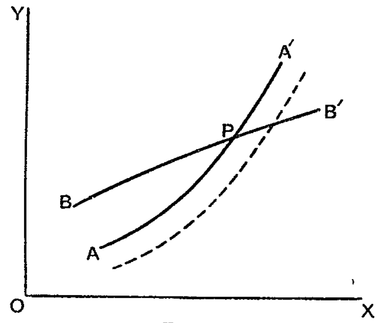

FIG. 5.

In other words, $\frac{\xi}{\eta}$ is not greater than $\frac{D_2}{D_1}$. This limit may be expressed by a line through the origin such as Oq in Fig. 4.

A line not passing through the origin may represent the condition that the *difference* between the two prices should not exceed a certain maximum.

(*b*) Corresponding propositions may be demonstrated for articles of which the production is complementary. We have simply to change the sign of $\frac{d^2\varphi}{dx\,dy}$; and accordingly the inclination of the price-curves, which will now be inclined and related as AA' and BB' in Fig. 5. The displacement caused by a tax on the x commodity is represented by a dotted curve on the right of AA' (on the understanding that the axes represent prices). Whence it appears that a tax on one of the complementary articles will cause the price of *both* to rise.

So far, supposing that the consumers of x and y constitute two distinct classes.[1]

Let us now suppose that the separation of classes no longer exists; and first let us suppose that x and y are quantities of rival commodities offered on a single market. For instance, x and y may denote respectively travelling on a railway by first or second class. (For simplicity we may suppose that there are only two classes, as commonly now in England; though they are commonly called first and *third*). This case is analogous to the preceding in so far as a tax on one commodity diminishes the quantity thereof which will be put on the market. But it does not now follow that the consumers of the substitute will be benefited. The consumers *in globo*, for instance, travellers on the railway as a body, may be prejudiced by a tax on one of the commodities, say travelling by first class; but it is also possible that they should be *benefited* thereby. The first proposition is self-evident; the second is a paradox which can only be demonstrated with the aid of mathematics.

Let $f_1(x,y)$ be the price of the first commodity when x and y are the quantities of the respective commodities that are taken by the market. Then $\frac{df_1}{dx}$ is, of course, negative. Also $\frac{df_1}{dy}$ is negative, since the increased consumption of x diminishes the demand for the substitute y. Let $f_2(xy)$ likewise represent the price of y. Then for the total net profit of the monopolist we have

$$V = xf_1(x,y) + yf_2(x,y) - \varphi(x,y).$$

For Δx and Δy, that is, the increments of the commodities due to a small tax τ on x, we have as before

$$\Delta x \frac{d^2V}{dx^2} + \Delta y \frac{d^2V}{dx\,dy} = \tau$$
$$\Delta x \frac{d^2V}{dx\,dy} + \Delta y \frac{d^2V}{dy^2} = 0.$$

From which it appears as before that $\Delta x \left(= \tau \frac{d^2V}{dy^2}\middle/ D\right.$ where D is positive) is negative. But Δy may be either positive or

[1] An important variety of this case occurs when a monopolist fixes different prices for the same article as consumed by different classes; for instance, a ticket for a theatre may bear a different price according as it admits a soldier or a civilian, a man or a woman. Many interesting examples of this type are adduced by Neumann in Schönberg's *Handbuch* (see an example given by *Dupuit*, below II. 404).

negative. All that can be said with certainty about its sign is that it will be contrary to the sign of $\frac{d^2V}{dx\,dy}$. But nothing is known about this second differential, except that it must satisfy the condition for V being a maximum, viz.—

$$\left(\frac{d^2V}{dx^2}\right)\left(\frac{d^2V}{dy^2}\right) - \left(\frac{d^2V}{dx\,dy}\right)^2 > 0;$$

a condition which does not depend on the *sign* of the quantity which is squared.* This condition is compatible with the supposition that not only should Δy be positive, but also $x\Delta\xi + y\Delta\eta$,[1] the approximate expression for the decrement of Consumers' Surplus, should be negative; that is, the consumers as a whole should be advantaged by the tax.

As, even with respect to mathematics, "seeing is believing," I subjoin a numerical example of the special case.** Let x and y be the quantities of two commodities which are rivals in consumption (partial substitutes for each other). Let the law of demand of these commodities be as follows, p_1 and p_2 being the respective prices

$$p_1 = 1{\cdot}605\dot{3} - {\cdot}2x - \tfrac{2}{3}\,(x - {\cdot}96)^{\frac{3}{2}} - \tfrac{1}{2}\,y$$
$$p_2 = 3{\cdot}9\dot{1}\dot{8} - 2(y - {\cdot}6975)^{\frac{1}{2}} - \tfrac{1}{2}\,x$$

for values of x and y in the neighbourhood of the vaules $x = 1$ and $y = 1$. This is a rational supposition, since there exists a function U such that $\left(\frac{dU}{dx}\right) = p_1$ $\left(\frac{dU}{dy}\right) = p_2$; and U is suited

* Nothing can be learnt about the sign in question from the laws of utility, since they tell us only that, if U is the Total Utility or Consumers' Surplus (*cp.* note to p. 117), $\frac{dU}{dx} > 0$, $\frac{dU}{dy} > 0$, $\frac{d^2U}{dx^2} < 0$, $\frac{d^2U}{dy^2} < 0$, $\frac{d^2U}{dx^2}\frac{d^2U}{dy^2} - \frac{d^2U}{dx\,dy} > 0$; whereas $\frac{d^2V}{dx\,dy}$ involves *third* differentials of U, about which nothing is given.

[1] The total net utility accruing to the consumers or the Consumers' Surplus obtained from the purchase of tho quantities x and y at the prices ξ and η respectively, may be written $U - x\xi - y\eta$; where U is identical with $F(x_1y)$, as defined on p. 117 above. The total net utility when $x + \Delta x$ is substituted for x and for y $y + \Delta y$, becomes (approximately)

$$U + \Delta x\left(\frac{dU}{dx}\right) + \Delta y\left(\frac{dU}{dy}\right) - (x + \Delta x)(\xi + \Delta\xi) - (y + \Delta y)(\eta + \Delta\eta)$$
$$= U - x\xi - y\eta + \Delta x\left(\frac{dU}{dx} - \xi\right) + \Delta y\left(\frac{\Delta U}{dy} - \eta\right) - (x\Delta\xi + y\Delta\eta).$$

Whence the increment to the total net utility due to the increments of x and y is $-(x\Delta\xi + y\Delta\eta)$ (since $\left(\frac{dU}{dx}\right) = \xi$ and $\left(\frac{dU}{dy}\right) = \eta$).

** The numerical data here used are not exactly the samo as those given in the example as originally set forth in the *Giornale*. The figures are now taken from a simplified version of that example (presented below, **F**, p. 148). Further fortification of the theory is offered at II. 93, **S**; and a fresh example at II. 400, ζ.

to represent the total utility (above a certain minimum) derived from the possession of the quantities of x and y distributed as described above (p. 117). For as $\left(\frac{dp_1}{dx}\right) = \left(\frac{d^2U}{dx^2}\right)$ is negative in the neighbourhood of the values $x = 1$, $y = 1$ (for which values $\left(\frac{dp_1}{dx}\right) = -\cdot 4$), there is, as there ought to be, a limit to the quantity of x which the consumers will take at that price, supposing the price of y to be fixed. There is a corresponding limit to the increase of y since $\left(\frac{dp_2}{dy}\right)$ is negative ($= -1\cdot\dot{8}\dot{1}$). Further, supposing both quantities to vary simultaneously, there is, as there ought to be, a limit to the amount of *sandwiches* of the form $lx + my$ which the consumers at any assigned (pair of) prices will demand; since the remaining condition for U being a maximum holds good, for $x = 1$, $y = 1$ (and in the neighbourhood), viz.—

$$\left(\frac{d^2U}{dx^2}\right)\cdot\left(\frac{d^2U}{dy^2}\right) - \left(\frac{d^2U}{dx\,dy}\right)^2 > 0.$$

Such being the laws of demand, we have for the monopoly profit $V = xp_1 + yp_2$, *i. e.* supposing at first that there are no expenses of production; which is a maximum when $x = 1$, $y = 1$, since then

$$\left(\frac{dV}{dx}\right) = 1\cdot 605\dot{3} - \cdot 4x - \tfrac{2}{3}(x - \cdot 96)^{\frac{3}{2}} - x(x - \cdot 96)^{\frac{1}{2}} - y = 0,$$

$$\left(\frac{dV}{dy}\right) = 3\cdot 9\dot{1}\dot{8} - 2(y - \cdot 6975)^{\frac{1}{2}} - y(y - \cdot 6975)^{-\frac{1}{2}} - x = 0;$$

while the second differential coefficients of V fulfil the remaining condition for a maximum; for

$$\left(\frac{d^2V}{dx^2}\right) = -3\cdot 3$$

$$\left(\frac{d^2V}{dy^2}\right) = -\cdot 6311$$

$$\left(\frac{d^2V}{dx\,dy}\right) = -1$$

$$(-3\cdot 3) \times (-\cdot 6311) - 1^2 = 1\cdot 0826 > 0.$$

If now a small tax of τ per unit is imposed on the first commodity we have for the increments of quantity (above, p. 131)

$$-3\cdot 3\,\Delta x - \Delta y = \tau$$

$$-\Delta x - \cdot 6311\,\Delta y = 0.$$

Whence

$$\Delta x = -\tau\cdot 6311 \div 1\cdot 0826 = -\cdot 5829\tau;\ \Delta y = 1 \div 1\cdot 0826 = +\cdot 9237\tau.$$

Accordingly the *decrement* of Consumers' Surplus

$$= x\Delta p_1 + y\Delta p_2 \text{ (approximately)}$$
$$= \Delta x\left(x\frac{dp_1}{dx} + y\frac{dp_2}{dx}\right) + \Delta y\left(x\frac{dp_1}{dy} + y\frac{dp_2}{dy}\right)$$
$$= -\cdot 583\,(-\cdot 4 - \cdot 5)\tau + \cdot 9237\,(-\cdot 5 - 1\cdot\dot{8}\dot{1})\tau$$
$$= -1\cdot 626\tau.$$

Since then the *decrement* of the *Consumers' Surplus* is negative, there is a positive increment of advantage to the consumers in consequence of the tax. Or is it easier to say that as *both* the prices are reduced, the purchasers must be gainers? *

The conclusion becomes *a fortiori* when there are expenses of production; for then we have at our disposal more functions with which to manipulate a favourable example.**

Thus a tax on first-class tickets may have the effect of lowering the fares for both first and third class, and so benefiting passengers in general. The number of travellers by first class will, however, be diminished notwithstanding the attractions of a lower fare; the counter attractions of the lowered second-class fares predominating.

The paradox which has been exhibited is presented by many other kinds of taxation, or more generally governmental regulation relative to commodities that are correlated in consumption. The correlated commodities need not be *rivals*, as in the preceding example; they may be *complementary*, such as the carriage of a passenger's luggage and the carriage of the passenger himself. Likewise a bounty on one of the correlated commodities may prove *injurious* to the consumers.*** Again, the limitation of the monopoly profit to a fixed percentage of the cost (including interest on capital) is not necessarily advantageous to the consumer. For the problem is then to maximise V subject to the condition that **** $V \ngtr i\varphi(x, y)$, where i is a given fraction. Then beginning with the case in which the fraction i is such that the limitation is only just beginning to be operative, we shall find as before that the variations in the Consumers' Surplus consequent upon the limitation depend upon the sign of the magnitude $\left(\frac{d^2V}{dx,dy}\right)$ (or the corresponding second differential coefficient with respect

* $\Delta p_1 = -\cdot 229\tau$, $\Delta p_2 = -1\cdot 4\tau$.

** The example is modified so as to illustrate this point in the article dated 1899, which is republished below, F, p. 149; where the ξ and η are used in the same sense as x and y in the present context.

*** Not stated explicitly in the original.

**** The symbol $\ngtr$ (not greater than) expresses the limitation better than the symbol = used in the original.

to ξ and η); which in general is not given. The principle applies very generally to the taxation of correlated consumable articles in a regime of monopoly.*

These paradoxes may be somewhat diminished by the use of a principle which Economics is entitled to borrow from the kindred science of Probabilities, or the "Art of Conjecturing." This is the presumption that in certain cases a quantity of which we do not know the sign may be treated as zero. In the leading case before us, if, as before, $f_1(x_1y)$ is the price of the first commodity and $f_2(xy)$ that of the second (when the quantities x and y are taken by the market), $\varphi(x_1y)$ is the total cost of producing x and y, and V the net advantage of the monopolist, we have

$$V = xf_1(xy) + yf_2(xy) - \varphi(xy);$$

$$\frac{d^2V}{dx\,dy} = \left[\left(\frac{df_1}{dy}\right) + \frac{df_2}{dx}\right] + \left[x\frac{d^2f_1}{dx\,dy} + y\frac{d^2f_2}{dx\,dy}\right] - \left[\frac{d^2\varphi}{dx\,dy}\right].$$

Of the three parts or terms of this expression (distinguished by square brackets), we know that the first is positive or negative according as the demand is complementary or rival; and that the third (with its sign) is positive or negative according as the production is complementary or rival. But we do not know the sign of the second term; and are therefore perhaps justified in ignoring it; especially when the sum of the two other terms is considerable, as may well be if production and consumption are either both rival or both complementary.

So far in this section we have been supposing that the monopolist, true to his etymology, is only a *seller*. But the method which has been indicated may readily be extended to the case in which the monopolist is the *buyer* of two or more correlated commodities. Thus it is possible that he may have a rival or complementary demand for goods supplied by distinct groups of producers. Or he may purchase goods of which the production is rival or complementary. And at the same time he may have a rival or complementary demand for those goods.

We may form any number of combinations with the attributes of which the properties have been deduced; always excepting those cases in which two or more monopolists are in the field.

SECTION IV.—In conclusion I now propose to restate in plain

* The exposition in the original is interrupted by the statement of two well-known or obvious propositions: (*a*) A progressive (as well as a simply proportional) tax on the profits of the monopolist does not affect the consumers (even in the case of correlated consumption). (*b*) The effect of limiting the monopoly profit to a fixed amount is indeterminate; consumers may be either benefited or prejudiced by the limitation.

words the principal results of the preceding mathematical analysis. They consist, as might be expected, rather in general views than in particular rules.

1. One of the principal uses of mathematics to the economist is, in the words of Professor Marshall, "to make sure that he has enough and only enough premisses for his conclusions (*i. e.* that his equations are neither more nor less in number than his unknowns)." This criterion applied to monopoly shows that frequently—I think it may almost be said normally—there is not a sufficient number of conditions to render economic equilibrium determinate, in the general case of a system of bargains in which more than one monopolist takes part. This may be affirmed with peculiar confidence in the case where two or more monopolists who are in competition deal with a great number of customers who also are competing with each other; for example, two railways which ply between the same points. The instability is due not merely to the hope of one monopolist to ruin a rival by "cutting prices," a case that has often been described; but also to a more fundamental, though less obvious cause. The instability does not cease in cases where it is not possible for one monopolist to drive the other completely off the field. Such might be the case if workmen of two nationalities—say Anglo-Saxons and Chinese—united respectively in two combinations, had to deal with competitive entrepreneurs, or with foreign customers. The proposition clearly stated by Cournot,[1] and to all appearance generally admitted, that in such a system the action of economic forces would tend to a definite position of equilibrium, a determinate set of values,—this plausible proposition is proved to be unfounded. In the regime of competition, as Mill or someone has said, things are always seeking their level. It is not so in the regime of monopoly.

The character of perpetual instability may likewise be affirmed of conditions in which the two competing monopolists deal not in identical, but rival, articles; for example, in the cases just now instanced it may be supposed that the services of the two railways, or the work of the two nationalities, though not quite identical, are capable of acting as more or less perfect substitutes for each other.

This theory is less evident, the opinion of Cournot is more plausible, in cases where the competing monopolists are dealing

[1] "Il est bien évident que dans l'ordre des faits réels et lorsque l'on tient compte de toutes les conditions d'un système économique, il n'y a pas de denrée dont le prix ne soit complètement déterminé."

not in rival but "complementary" articles; for example, if the rolling-stock of a railway were possessed by one company and the railway-stations by another, or if the common labour necessary for the production of an article were monopolised by one combination, and the more highly skilled work without which the manual labour would be useless by another combination. Professor Marshall seems to contemplate this case when he supposes that a mill belongs to one monopolist and the water for driving it to another.[1]

Let us suppose that the two lettings are yearly; beginning at the middle of the year for the mill, and at the end of the year for the water-supply. If at midsummer the owner of the premises, when renewing his contract with the lessee, estimates what such a one can pay, on the basis of what he pays and will pay for the next six months for the use of the water—if the owner of the mill ignores the possible action of the owner of the water at the end of the year—then perhaps the reasoning of Cournot in a similar case will hold good. There will be a determinate equilibrium characterised by the curious property that the tenant will be worse off than if both had belonged to the same individual. That is, supposing that there are a number of mills at the disposal of the landlord, and a number of millers competing with each other.

But ought we to suppose that the proprietor, when renewing his contract, does not take into consideration possible future events? Will he not, theoretically, fix the rent at that figure which will be the most advantageous for him *in view of the rent which the owner of the water-supply may fix the next winter?* It is thus that a chess-player when making his move takes account of the move which his adversary will probably make. And, as in chess, when only the two kings and one of the inferior pieces remain on each side, may not the two monopolists go on making moves against each other to all eternity?

Those who adhere to Cournot's reasoning may be confronted with the supposition that one of the two monopolised articles, for instance, the water-power in the above example, passes into the hands of competitors. There will then be a regular market for water-power, offered by the competing owners to competing millers. Accordingly, given the rent of the mill, the payment for water-power will be determined by the usual equation between demand and supply (the total supply of water-power may be supposed a fixed quantity). According to the opinion

[1] *Principles of Economics*, third ed., Book V. ch. x.

here disputed, when once the charge for water-power has been settled by the market, the monopolist will treat this price as something sacred, and will only vary the rent of his premises *subject to the condition that the charge for water-power should not be disturbed.* But surely the general rule is that he will continue to vary the price of the monopolised article as long as that price multiplied by the quantity sold at that price—less the cost of production—continues to increase. It does not matter to him that the customers, in view of his changing that price, are obliged to modify their bargains with a third party. What difference can it make to the motives of the monopolist that the third party consists of a monopolist, not of individuals competing against each other? In both cases, indifferent to the interests of the third party, he will vary his price by successive steps in the direction which promises him an increase of profit. The only difference between the cases is that when the third party consists of competitors, a definite position of equilibrium will be reached (the tentatives of the single monopolist must come to a stop, or at least hover about a determinate point); whereas when the third party consists of a second monopolist, the conditions which bring about the equation of demand and supply in a competitive market are wanting. That is, excepting the arbitrary supposition that the second monopolist is such a fool as to act in the manner ascribed to him by Cournot's equation. But even if he were to do so, though there would exist a definite position of equilibrium, it would not be the one assigned by the theory here combated.*

This theoretical difference between the regime of monopoly and that of competition may have some bearing on practical issues, affecting as it does our views about trade unions and similar combinations. I have seen it proposed as an economic ideal that every branch of trade and industry should be formed into a separate union. The picture has some attractions. Nor is it at first sight morally repulsive; since, where all are monopolists, no one will be the victim of monopoly. But an attentive consideration will disclose an incident very prejudicial to industry —instability in the value of all those articles the demand for which is influenced by the prices of other articles; a class which is probably very extensive.

Among those who would suffer by the new regime there would be one class which particularly interests the readers of this Journal, namely the abstract economists, who would be

* It would correspond to the point Q, not to P, in Fig. 3.

deprived of their occupation, the investigation of the conditions which determine value. There would survive only the empirical school, flourishing in a chaos congenial to their mentality.

2. Professor Marshall exemplifies another use of mathematical reasoning when by means of his curves he demonstrates that it might be advisable to tax one kind of commodity and employ the proceeds in bountying another kind.[1] The abstract reasoning serves as a corrective to what has been called the " metaphysical incubus " of dogmatic *laisser faire*. In the case of monopoly indeed this incubus has not been serious : " it has never been supposed that the monopolist in seeking his own advantage is naturally guided in that course which is most conducive to the well-being of society regarded as a whole." Nevertheless, in so far as something similar to the old doctrine of economic harmony seems to be reappearing among the apologists for railway administration, a certain interest may attach to propositions unexpectedly favourable to the intervention of Government in businesses subject to monopoly. Such is the proposition above proved, that when the supply of two or more correlated commodities—such as the carriage of passengers by rail first class or third class—is in the hands of a single monopolist, a tax on one of the articles—*e. g.* a percentage of first-class fares—may prove advantageous to the consumers as a whole. Thus in the instance given the advantage would accrue not only to those who before the tax travelled third class, and continue to do so afterwards, but the travelling public in general, including first-class passengers. The fares for *all* the classes might be reduced.

3. To obtain rules directly applicable to practice there would be required a knowledge of concrete details beyond what the present writer can command. Still, some suggestions bearing on the control of monopolies by governmental interposition may be derived from the preceding analysis.

A first step in this direction was made by Cournot when he proved that a tax of an ordinary kind on a monopolised product has the effect of increasing the price. This is contrary to the judgment of some distinguished writers who hold that, the monopolist having already done his worst against the customer, the burden of the latter cannot be increased by a tax. There is, however, a limiting case in which the popular opinion is correct ; namely, where a monopolist buyer deals with sellers of an article which is absolutely limited in quantity (land, for instance), or can only be increased with great difficulty. A building syndicate

[1] *Principles*, Book V. ch. xii. pp. 555–7. *Ibid.* ch. xiii. (third ed.)

buying up land from uncombined owners may afford an example.

Cases specially favourable for the application of mathematical analysis occur where we have to deal with *correlated* (connected) supply and demand. Suppose, first, the supply only connected, as when a railway company, the greater part of whose expenditure (interest on capital, cost of repairs, etc.) cannot be attributed exclusively to one branch of the business, serves two classes of customers whose interests are quite separate, say traders requiring their goods to be carried and passengers other than commercial travellers. Here we must distinguish two classes: (*a*) *complementary* products, in the case of which the production of one article becomes less difficult and expensive by the increased production of the other article ("joint" products as defined by Mill are included in this class); (*b*) *rival* products, in the case of which the production of one article becomes more costly according as the production of the other is increased. The first case usually occurs where the law of increasing returns rules; for instance, if the general expenses of a railway do not increase in proportion to the traffic, the increase of one kind of traffic tends to make the increase of the other kind more remunerative (see above a more exact definition). Contrariwise, when the land or the capital at the disposal of the company is fully occupied, it is possible that the increase of one service may render another less profitable than it would otherwise have been. The proprietors of a railway with only one or two tracks may find that the increase of the goods traffic causes the passenger traffic to be attended with greater expense; the fuel of the company and the labour of its employees being wasted while the passenger trains have to wait in side tracks to avoid collisions.

It is very possible that both tendencies may be present, not coincidently, but with reference to a different extent of variation in the products under consideration. Thus a certain increase in the goods traffic by crowding the present line as above described might act in *rivalry* to the passenger traffic; but with reference to a large increase in the goods traffic, such as to make it profitable to have an additional track and so obtain the economies of production on a large scale, the goods traffic may be considered as *complementary* to the passenger traffic.* I do not pretend to discern to which of the two categories each concrete case belongs; I only wish to distinguish their properties in the abstract.

Among methods of governmental control, one of the most

* See Index, *sub voce* Joint Production.

important is that which consists in fixing a maximum tariff; provided that the maximum is not suspended on high, but is such as really to restrain the action of the monopolist; in which case its operation is nearly similar to that of a fixed tax. Suppose now that the price of one product is fixed, but not so that of another; or, what is more probable, that there is an effective maximum for one article and an inoperative maximum for another. The effect on the price of the second article will differ according as the products are *complementary* or *rival*. If they are complementary, the lowering of the price of the first is followed by the lowering of the price of the second; the benefit in respect of one commodity is a benefit also in respect of the other commodity. If the products are rival, there is a benefit to one class of consumers and a loss to another; provided, of course, that the loss to the monopolist is not so great as to induce him to give up the business.

The same rule applies to the effects of a law which requires that the price of an article in one market should not exceed its price in another by more than a certain percentage.[1] What is a benefit in respect of one commodity will be also a benefit in respect of the other, if the products are complementary; but a loss in the case of rival production.

A corresponding rule applies to a tax of the kind called "specific," that is, of so much per unit of commodity. The loss to one class of consumers will be a loss to the other class in case of complementary products; but a gain in the case of rival products.

The case of connected *demand* does not admit of equally definite rules. It is probable, but not certain, that the rules enounced for rival and complementary production hold good respectively for complementary and rival demand. Thus a maximum which lowers the rate for the terminal services of a railway tends probably to raise the rates for carriage since the demands for the two services are complementary. But a maximum which lowers the fare for third-class passengers tends probably to lower the fares for the first class, since the demands for the two kinds of tickets are rival.

The probability increases when the tendency of demand is in the same direction as that of production, and diminishes in the contrary case.

These propositions respecting the influence of demand may

[1] Generalising the conceptions of the American "*Short-haul Clause*," as it is commonly understood.

be applied to a law against differential charges and to a specific tax.

A tax proportional to the profits of the monopolist falls entirely upon him, as Cournot and Professor Marshall have proved. It should be added that a "progressive" tax on monopoly profit acts similarly.[1]

The effect of limiting the profit of the monopolist to a fixed amount is generally indeterminate. It may be advantageous or detrimental to some or all or none of the various groups of his customers. The fixing a (*bonâ fide*) maximum rate of profit on the capital expended acts to the advantage of the consumer.

I am not blind to the practical difficulties which stand in the way of a tax on the net profits of a monopolist, and of other measures that are here discussed. It cannot be too often repeated that the rules derived from mathematical reasoning are essentially abstract and require in practice to be largely diluted with common sense.

[1] Since this article was printed I have found that Knut Wicksell had preceded me in pointing this out.