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PROFESSOR SELIGMAN ON THE THEORY OF MONOPOLY

[THIS article, which appeared in the ECONOMIC JOURNAL, 1897, under the title " Professor Seligman on the Mathematical Method in Political Economy," might, in accordance with that title, equally well have been placed in the mathematical section of this Collection. The subject-matter is the theory of monopoly; the form is largely mathematical. The mathematical method is tested by an encounter with the classical method wielded by a powerful hand. The victory, if indeed it has been won, is of a somewhat Pyrrhic character. For it does not much redound to the credit of the mathematical method that the points in which it has an advantage are just those which have been neglected by an economist of conspicuous wisdom, one whose judgments on Income Taxes, Public Loans and other momentous interests are generally approved and followed. It is strongly suggested that matters which such an author did not take pains to state precisely cannot be of great importance. They might be compared to the points of detail on which the critical shoemaker corrected the masterpiece of the Grecian painter. Even without those little corrections the piece would no doubt have been a first-rate work of art. So the Art of Political Economy is not much affected by judgments on the question whether the taxation of a monopolised article is likely to be more or less burdensome to the consumer according as the production obeys the law of increasing or decreasing returns, or according as the demand is more or less elastic. Still, if such questions are posed, it is better not to answer them carelessly.

The criticisms on the original version having been directed against the second edition of Professor Seligman's *Shifting and Incidence*, published 1899, some of them have now been withdrawn or modified in deference to emendations in the third edition, published 1910.]

I. (1) Following the order in which Professor Seligman has discussed the several issues, I notice first his objection to my theorem that if a monopolist deals in two commodities for which there is a rival demand, such as first- and third-class service in a railway like our Midland with only those two classes of passenger fares, then if a tax is imposed on first class fares, it is theoretically possible that the monopolist proprietary of the railway may find it to their interest to *lower both* fares. Upon this Professor Seligman remarks :—

" That it [the mathematical method] sometimes leads to results which are likely to divorce still more the economics of the closet from the economics of the market-place, may be illustrated by a slip of Mr. Edgeworth himself. [See the extended mathematical proof (in the ECONOMIC JOURNAL, vii., pp. 230–232) of the proposition that a tax on first-class railroad tickets will reduce (not increase) the price of tickets of *all* classes]. The mathematics which can show that the result of a tax is to cheapen the untaxed as well as the taxed commodities will surely be a grateful boon to the perplexed and weary secretaries of the Treasury, and ministers of finance throughout the world." [1]

I hope that a sufficient reply will be made to these objections if (α) I show that the proposition in question is agreeable to *a priori* presumptions, and (β) I verify it by a numerical example.

(α) The presumptions of the case are summed up in an answer which I received from an eminent economist—not specially devoted to the mathematical method—to whom I had submitted my little theorem, asking if it seemed to him too paradoxical to be credible. " I should hardly be surprised at anything in a regime of monopoly," was his reply. I might illustrate the paradoxical character of this regime by referring to one of its peculiarities which is relevant to the present question. If the demand for an article is *raised* [2] in the sense that more of it is demanded at each price than before; then, whereas in a regime of competition, *ceteris paribus*, theoretically in general the price will rise, this rule is not equally universal in a regime of monopoly : there the price may fall while the demand rises.[3]

The unexpected is all the more likely to occur in the case before us in that it is a case, not only of monopoly, but also of rival demand. Even in a regime of competition, as I have pointed out,[4] the taxation of articles for which the demand is *correlated* (either rival or complementary) is apt to produce curious results. *A*

[1] *Shifting and Incidence*, 2nd edition, p. 174.

[2] As to this technical use of the term rise of demand see Sidgwick's *Political Economy*, II. ch. ii. § 2, and *cp.* Marshall, *Principles of Economics*, Book III. ch. iii., Art. 4, Book V.

[3] See ECONOMIC JOURNAL, Vol. VII. p. 234.

[4] *Ibid.*, Vol. VII. p. 55.

fortiori, when the peculiarities of monopoly are combined with those of correlated demand. In the case of independent demand for a single article, in a regime of monopoly, the consumer may bear only a very small [1] proportion of the tax, even under the law of constant (not to say increasing) cost, under which the proposition would not be true in a regime of competition. When the circumstance of rival demand is superadded to monopoly, is it to be wondered at that the abnormality, as it appears in relation to the simpler case usually contemplated, should be increased: that the consumer should not only not be damnified, but should even be somewhat benefited by the tax?

A general idea of the modification due to the introduction of rival demand may be obtained by observing that alike in the case of independent and that of correlated demand, a tax on a monopolised article results in the diminution of the quantity of that article put on the market; [2] but, while in the case of independent demand the diminution of the quantity supplied is attended with a rise of price, this consequence does not necessarily follow in the case of rival demand. A tax on first-class fares results in the diminution of the quantity of first-class service supplied; and accordingly there is a flow of passengers from first class to third class. The demand for third-class service thus *rises* in the technical sense referred to in a former page, where it was stated that in monopoly this rise of demand may be attended with a fall of price.[3] The lowering of third-class fares results in a *fall* of the demand for first-class service in this technical sense, that for every possible first-class fare, the third-class fare being supposed constant at its new figure, the amount of first-class service demanded is less than what it would have been for each first-class fare before the disturbance, the third-class fare being supposed constant at its old figure. Now, it is quite consistent with ordinary presumptions that, when the demand for an article falls in this sense, its price should fall. Accordingly, the first-class, as well as the third-class fare, may be reduced.[4] *A fortiori*, it is possible that,

[1] The argument considered as *ad hominem* becomes *a fortiori*, since Professor Seligman thinks it possible that the consumer in this case may bear *no* part of the tax (*cp.* below, p. 161).

[2] I suppose that this proposition would be accepted by an opponent, as it is what may be expected from the analogy of competition, and is less than what those who trust that analogy accept. For a proof of the proposition I must refer to my article on "La Teoria pura del Monopolio" in the *Giornale degli Economisti* for 1897. [Above, E].

[3] Above, p. 144.

[4] It may assist conception to imagine first-class and third-class services controlled by different managers. The steps described in the text might be made by

though the first-class fare may be a little increased, yet the third-class fare may be so much diminished that the consumers as a whole may gain.

Considered as a mere possibility, this statement is not open to Professor Seligman's raillery in the passage above quoted. The plausibility of his objection is obtained by substituting a downright indicative—" the tax *will* reduce . . . the price" . . . " the result of the tax *is* to cheapen" . . .—for the potential mood in which I had couched my proposition, " a tax on one commodity *may* benefit the consumers of both . . . " the consequences of the new tax *may* be," . . . and so on.[1] I added the following caution :—

" Of course I do not suppose so delicate an adjustment—such a frictionless movement towards the position of maximum profit—to be realised in the concrete management of an English railway. But I think that it may be of scientific interest to establish the theoretic possibility of the paradox." [2]

The proposition in question is to be taken in the same spirit as the paradox of Mill, that an *improvement* in the production of an export may be *detrimental* to the exporting nation.[3] What should we think of a free trade writer who remarked on Mill's theorem that it would surely be a grateful boon to weary and perplexed ministers of commerce, since now all they had to do in order to foster commercial prosperity would be to injure the manufacture of exported commodities ! Mill's theorem is useful as presenting an extreme and striking instance of a general truth which, if not indeed paradoxical, is yet not so familiar, but that it is desirable to call attention to it, the important truth that the interests of the parties to international trade are not so completely identical as some free traders have supposed. So our paradox calls attention to the truth that taxation in a regime of monopoly is more diversified and irregular in its consequences, and I think it may be added, likely to be less detrimental to the consumer, than an equal impost in a regime of competition. The extreme exemplifications of these truths are not designed to ease " perplexed and weary ministers," but to startle indolent and prejudiced

the respective managers each endeavouring to maximise the net return in his department. But the further step, which on this supposition would be likely to occur, namely, the continued reduction of the first-class fares in order to steal custom from the third-class department, would be stopped by the directors of the common concern, who would not allow a gain to the first-class department to be purchased by a preponderant loss to the third-class department.

[1] ECONOMIC JOURNAL, Vol. VII. pp. 230 and 232. [2] *Ibid.*, p. 231.

[3] *Political Economy*, Book III. ch. xviii. § 5.

economists from their dogmatic slumber, and incite them to reflect that maxims learnt too well from the study of familiar cases cannot always be applied without modification beyond the sphere of experience on which they were founded.[1]

These preliminary considerations will, I hope, dispose the student to attend to the mathematical ratiocination by which I have elsewhere deduced the theorem under consideration.[2] Addressing the general reader rather than the mathematician at present, I will not repeat this second part of the proof. I confine myself to the third stage, the verification, which consists in instancing laws of cost and of demand which actually fulfil the theory.

(β) Let us put p_1 as the price of a first-class ticket for a certain journey, or number of miles, and p_2 as the price of a third-class ticket for the same journey. At these demand-prices let the number of the first-class passengers be x, that of the third-class passengers be y. Then, agreeably to the general laws of demand, p_1 must be so related to x that, other things being the same, p_1 decreases as x increases (and conversely); and p_2 must be similarly related to y. Also as first-class and third-class service are *rival* commodities, an increased supply of third-class service, while the amount supplied of first-class service remains constant, will be attended with a decrease in the first-class fare at which that amount of first-class accommodation is demanded. And the numbers of the first-class passengers will be similarly related to the third-class fares. These conditions, and any others that may reasonably be required,[3] are fulfilled

[1] On the meaning and use of *paradoxes*, compare De Quincey, *Words*, Ed. 1889, i. p. 199, and vii. p. 206. "Several great philosophers have published, under the idea and title of paradoxes, some first-rate truths, on which they desire to fix public attention, meaning, in a short-hand form, to say: 'Here, reader, are some extraordinary truths, looking so very like falsehoods, that you would never take them for anything else if you were not invited to give them a special examination.'"

[2] *Giornale degli Economisti*, 1897. [Above, E.]

[3] Professor Irving Fisher in his *Mathematical Investigations* has suggested the question whether the prices of two articles, x and y, for which the demand is correlated, must be regarded as the partial differentials, with regard to x and y respectively, of a certain function which represents the total utility afforded by any quantities of x and y. I answer this question in the affirmative (See *Giornale degli Economisti*, 1897, p. 314 note), with the same reservations as the conception of total utility requires in the case of a single variable, in particular that it should not be measured from the extreme point of privation: and accordingly take $p_1dx + p_2dy$ in my example as a complete differential of a function which I need not write out, but may call U. It may facilitate conception to consider the case in which x and y are not articles of consumption but factors of production; for instance, the carriage of different kinds of goods, for which p_1 and p_2 are the respective fares. Then U may stand for the sum of the producers' surpluses

over a considerable range of prices [1] by the following laws of demand :—[2]

$$p_1 = £5 \times \left(1\cdot605\dot{3} - \frac{\cdot 2x}{20,000} - \frac{2}{3}\left(\frac{x - 19200}{20,000}\right)^{\frac{3}{2}} - \frac{1}{2}\frac{y}{100,000}\right)$$

$$p_2 = £1 \times \left(3\cdot9\dot{1}\dot{8} - 2\left(\frac{y - 69750}{100,000}\right)^{\frac{1}{2}} - \frac{1}{2}\frac{x}{20,000}\right)$$

We may begin by supposing the cost constant—a very possible case, as Cournot has remarked.[3] Then the profit, which it is the object of the monopolist to maximise, the net monopoly revenue, in the phrase of Professor Marshall, is of the form $x \times p_1 + y \times p_2 - C$, where C is a constant. It may be shown first that this net profit is a maximum, when $x = 20,000$, $y = 100,000$ corresponding to the fares $p_1 = £5 \times \cdot 9 = £4\ 10s.$; $p_2 = £1 \times 2\cdot3\dot{1}\dot{8} = £2\ 6s.\ 4\cdot\dot{3}\dot{6}d.$; secondly, that if there is imposed on first class travelling any tax of so much per ticket, not exceeding $\cdot 16885$ £5, or about 16*s.* 10½*d.* per ticket, it will become the interest of the monopolistic company to *lower* the fares *both* for first-class and third-class passengers.

The statement may be conveniently altered by putting ξ for $x \div 20,000$, and η for $y \div 100,000$. Then the expression which is to be maximised becomes $20,000\xi \times 5(1\cdot605\dot{3} - \cdot2\xi - \frac{2}{3}(\xi - \cdot96)^{\frac{3}{2}} - \frac{1}{2}\eta) + 100,000\eta\ (3\cdot9\dot{1}\dot{8} - 2(\eta - \cdot6975)^{\frac{1}{2}} - \frac{1}{2}\xi)$ (+ a constant). And it has first to be shown that this expression is a maximum when $\xi = 1$ and $\eta = 1$. The reader to whom this sort of investigation is not familiar may be advised to substitute in (the variable part of) the above-written expression values for ξ and η at first very close to unity, *e. g.* for ξ, $1\cdot001$, or $1 - \cdot001$, and for η values about equally distant from unity; then gradually enlarging the divergence from unity to realise that for a considerable distance on both sides of unity the result of substituting different values of ξ and η is to make the expression smaller than what it is for $\xi = 1$ and $\eta = 1$: that is, $3\cdot2\dot{1}\dot{8} \times 100,000$.[4]

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enjoyed by the customers of the railway on the assumption that each producer will push his expenditure on each factor of production up to the margin of profitableness.

[1] As to the range over which the formulæ are applicable, see below, p. 149, note [and ECONOMIC JOURNAL, Vol. XX. p. 291, in a note not reprinted here for a reason mentioned in the Introduction].

[2] From these simultaneous equations we can obtain x and y in terms of p_1 and p_2; as stated ECONOMIC JOURNAL, 1897, p. 54.

[3] *Principes Mathématiques*, Art. 30.

[4] If $\Delta\xi$ be the difference between the assumed value of ξ and unity, and $\Delta\eta$ the difference between the assumed value of η and unity, then, $\Delta\xi$ and $\Delta\eta$ being small, the increment of the monopolist's net profit consequent upon the change from ξ

Consider next the consequence of imposing a moderate tax of so much per ticket on first-class fares, say, £5 × τ, where τ is a fraction not exceeding ·16885. The amount to be maximised is no longer now $xp_1 + yp_2$, but the same $- 5\tau x$; that is, if we employ ξ and η as before, the same expression as before, *minus* $100{,}000 \times \xi\tau$. The value of this modified expression when $\xi = 1$ and $\eta = 1$ is $100{,}000\ (3{\cdot}2\dot{1}8 - \tau)$. It will be found that for any assigned value of τ (up to the limit mentioned), there can be found a value of ξ less than unity, and a value of η greater than unity, such that the monopolist's revenue, as modified by the tax, should be a maximum for those values, while *both* the prices—both the first-class and second-class fares—are *less* than what they were before the imposition of the tax.

Here, as before, the reader may be advised to begin with small quantities. To any small value of τ there may be expected to correspond two values of ξ and η in the neighbourhood of unity, rendering the (modified) monopolist revenue a maximum. For example, to $\tau = {\cdot}0017155\cdots \times$ £5, a little more than twopence per ticket, there correspond the values $\xi = {\cdot}999$, $\eta = 1{\cdot}0015845231\cdots$; and it may be found by actual substitution, or better by general reasoning,[1] that the loss to the monopolist through the decrease of his receipts is ·000,0009 × £100,000 nearly, while his gain in having to pay tax on a smaller number of first-class tickets is double that amount, viz. ·000,0017 × £100,000 nearly. The monopolist is, therefore, better off with

and η is approximately $-\frac{1}{2}(3{\cdot}3\Delta\xi^2 + 2\Delta\xi\Delta\eta + {\cdot}6311\Delta\eta^3)$; as may be shown by expanding the surd or irrational terms in the expression for the profit according to the *algebraic* rule for extracting the square root (*cf.* Todhunter's *Algebra for Beginners*, ch. 28), and neglecting powers of $\Delta\xi$ and $\Delta\eta$ above the second. The above expression for the increase of the profit is negative, whatever the *signs* of $\Delta\xi$ and $\Delta\eta$: showing that the profit corresponding to $\xi = 1$, $\eta = 1$ is greater than for any other values in the immediate neighbourhood. In whatever direction we step from the position defined by the equality of ξ and of η to unity, we descend and continue to descend to a considerable distance in every direction—for instance up to $\xi = {\cdot}96$, η remaining 1, up to $\eta = {\cdot}6975$, ξ remaining 1, and much further in the *positive* directions of ξ and η. The stoppage at those points has, of course, no economic significance: it was adopted merely for the sake of arithmetical convenience; otherwise it would have been better to use cube roots where now square roots are used.

[1] The clue to the investigation is given by the following equations. [See E, p. 127.] Let the new ξ (corresponding to the maximum after the tax) be $1 + \Delta\xi$ (where $\Delta\xi$ is a small quantity positive or negative), and similarly let the new η be $1 + \Delta\eta$. Then the values of $\Delta\xi$ and $\Delta\eta$ in term of τ are given by the following (simultaneous) equations

$$\begin{cases} \Delta\eta + 3{\cdot}3\Delta\xi + \tau = 0. \\ {\cdot}6311\Delta\eta + \Delta\xi = 0. \end{cases}$$

These equations are approximately satisfied by $\Delta x = -{\cdot}5829\tau$, $\Delta y = \tau/108263$, They become less and less exact as $\Delta\xi$ and $\Delta\eta$ are increased, with the increase of τ.

the new values of ξ and η than he would be (after the imposition of the tax) with the old values, and, as it will be found, with any other values of ξ and η. But these values of ξ and η correspond to *lower* values of p_1 and p_2 (first- and third-class fares) than existed before the tax; as may be seen if these values are substituted for ξ and η in the expressions for p_1 and p_2 respectively.

As we increase τ and therewith decrease ξ and increase η, the general relations which have been indicated persist: the monopolist gains more by escaping part of the tax than he loses by the diminution of the receipts, as the values of ξ and η move further away from unity (the proportion of the gain to the loss becoming greater as the absolute quantities become greater), while both the fares continue to diminish. Thus for the limiting value of τ, viz., ·16885·· we have $\Delta\xi = -\cdot 04$ and $\Delta\eta = \cdot 05248$. And by actually substituting ·96 for ξ, and 1·05248 for η, it is found that the new receipts are less than the old receipts by about ·002 × £100,000. Against this loss is to be set off the gain of saving the tax of £5 × ·16885 on ·04 × 20,000 first class tickets: that is a gain of ·00675 · · · × £100,000—a gain more than three times greater than the loss. At the same time the first-class fare is diminished by $(\frac{1}{2}\ \cdot 05248 - \cdot 2 \times \cdot 04 - \frac{2}{3}\cdot 008) \times$ £5, that is, diminished by ·0129 × £5 nearly, = 1*s*. 3½*d*.; and the third-class fare is diminished by £2 $(\sqrt{\cdot 3025 + \cdot 05248} - \cdot 55) - \frac{1}{2}\cdot 04$, = £·0716, = 1*s*. 5*d*. nearly.

The net gain of some £475, which we have found to attend the lowering of both fares, might well be a substantial percentage of the net profits, supposing these to be, say, about 7 per cent. of the gross profits, which were originally £321,818·18. Say the net profits (per month or year) are about £20,000, the monopolist gains about 2 per cent. on his net profits by making the adjustment described.

We have so far been supposing the total cost to be a fixed amount, say about £300,000. But the reasoning is not materially altered when we suppose the cost variable. To take a simple instance, let the cost consist partly of a constant sum, and partly of two additional amounts respectively proportional to x and y, the number of first-class and that of third-class tickets. To obtain now the expression for the monopolist's net revenue we must deduct from the gross receipts $xp_1 + yp_2$, the cost $xk_1 + yk_2$, where k_1 and k_2 are the cost per unit of x and y respectively. Much the same conclusion as before may be brought out if we put $p'_1 = p_1 + k_1$, $p'_2 = p_2 + k_2$; the expression which the monopolist seeks to maximise becoming now $xp'_1 + yp'_2$.

One consequence of admitting the variation of the cost is to render the occurrence of anomalies more probable. When correlation of cost is superadded to correlation of demand, and both to monopoly, we may look out for freaks in the incidence of taxation.*

I should be curious to know what " slip " has been detected in this reasoning, substantially identical with what has been already published.[1] In a matter of this sort imputations of error based upon first appearances should be objected sparingly.

(2) I take next Professor Seligman's theory as to the relation between the law of cost and the pressure of taxation.**

The debate on this matter has been somewhat embarrassed by the disputants having used the principal terms in different senses. According to the definition which I have employed in

* It will be shown on a later page (II. 124) that even in the regime of Competition, when both cost and demand are correlated, a tax on one of the articles may cause the price of *both* to fall. *A fortiori* of course in the regime of Monopoly.

[1] The example given in my article in the *Giornale degli Economisti*, 1897 [above, E], differs only in details from the one here given.

** There is here omitted the statement of a thesis maintained by Professor Seligman in the *second* edition of his *Shifting and Incidence*, viz. : " Under ordinary conditions, therefore, in the case of a tax on monopolistic industry subject to the law of increasing returns or diminishing cost, the tendency is that the consumer will suffer less than in the case of an industry subject to the law of constant cost. . . . On the other hand, if the monopoly obeys the law of diminishing returns or increasing cost, the producer will be likely to add more of the tax to the price than in the case of constant or increasing returns." This thesis being no longer maintained by the author (*third* edition, p. 248, note), I have withdrawn the lengthy dialectic which in the original article was directed against certain arguments used in the *second* edition in support of the thesis as stated there. But it is not necessary to withdraw all the negative reasoning in this context. The two paragraphs retained at the end of the section headed (2) are applicable, with a slight alteration, to the thesis of the *third* edition; which may be described in logical terms as the *contrary*, whereas it ought to have been only the *contradictory*, of the original thesis. According to the new thesis, " if a monopolised article is the product of an industry which obeys the law of increasing return or diminishing cost," then, " as in the case of competition, he [the monopolist] will add more of the tax to the price. *Vice versa* under conditions of diminishing return, or increasing cost, the monopolist producer will be likely to add less of the tax to the price than in the case of constant return " (*Shifting and Incidence, third* edition, p. 247). It will be here maintained that the true proposition respecting increasing and diminishing returns in the second of the senses above distinguished has by no means the universality expressed by the above statement. The positive portion of section (2) is retained partly as possessing, it may be hoped, some intrinsic interest, partly as leading up to the retained portion of the negative reasoning, and partly as relevant to the issue between diagram and algebra referred to on a later page (167). A glance at the formula there given reveals more quickly than does a study of the diagrams in this context the truth that the law of " returns " in our sense of the term does, and in the other sense does not, constitute an index of the pressure on the consumer due to an assigned tax.

the articles referred to, the law of increasing cost, synonymous with diminishing returns, holds good when the total cost of producing the quantity x of a certain commodity increases with the increase of x at an increasing rate; the law of diminishing cost, synonymous with increasing returns, holds good when the total expense of producing the quantity x increases with the increase of x at a diminishing rate.* In other words, the law of increasing cost, or diminishing returns, holds good when the ratio of the last increment of cost to the last increment of produce is greater than the ratio of the penultimate increment of cost to the penultimate increment of produce; with a corresponding statement for the law of diminishing cost (or increasing returns). "This definition," I intimated, "is not identical with that of some distinguished economists"; who may seem to compare the ratio of the last increment of cost to the last increment of produce, not with the ratio above stated, but with the ratio of the total expense to the total produce x, and may accordingly [1] define that the law of increasing or diminishing cost holds good according as the ratio of total cost to total product increases or diminishes with the increase of product.[2]

A geometrical illustration may put the matter in a clearer light.

In the accompanying diagram the ordinate Y of the curve O_1Q represents the total expense required to produce any amount, x, of a certain commodity, represented by the corresponding coordinate. The case represented is one in which a certain amount of expense, OO_1, must be incurred before any return at all is obtained. According to my definition—No. 1, we may call it—the law of increasing cost, or diminishing returns, holds good for all tracts of a curve oft his sort which are *convex* to the axis of X, that is, in the case illustrated, throughout. According to the other definition, No. 2, the law of increasing cost holds good only for those tracts for which the slope of the curve, the inclination of a tangent at any point of the curve to the axis of X, is greater than the inclination to that axis of a line joining that point to O. In other words, according to definition No. 2, the law of increasing cost holds good while the ratio of total cost to produce increases

* Above, E, p. 67 *et seq.*

[1] If x is the quantity produced and $f(x)$ the corresponding total cost, it comes to much the same whether we take as the essential attribute of increasing cost the fact that $f'(x)$ is greater than $f(x) \div x$, or that $f(x) \div x$ increases as x increases.

[2] There may be other shades of meaning, especially in the case of competition as distinguished from monopoly. The difficulties presented by "increasing returns" in a regime of competition are noticed in one of the articles referred to (S, II. 87 [*cp.* ξ]).

with the increase of produce. The relation between the two definitions is illustrated by the diagram. Beyond the point Q, at which a tangent to the curve passes through O, the law of increasing cost holds good in both senses; but on the near side of Q there is increasing cost in the first sense, but diminishing cost in the second sense. If the origin had been at O_1, the axis of X being a horizontal through that point, then the law of increasing returns would prevail *throughout* in the second sense as well as in the first sense. If any cost-curve possess either of the attributes continuously *ab initio* in one sense, then it will possess that attribute in the other sense also throughout.

The same diagram (which may with advantage be viewed from behind the paper) can be used to illustrate the different definitions

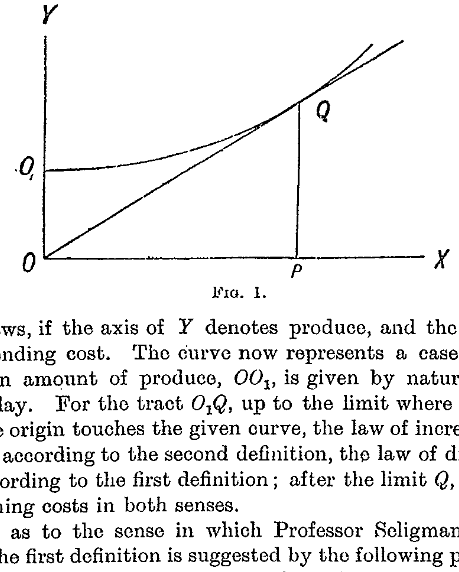

FIG. 1.

of the laws, if the axis of Y denotes produce, and the axis of X corresponding cost. The curve now represents a case in which a certain amount of produce, OO_1, is given by nature without any outlay. For the tract O_1Q, up to the limit where a tangent from the origin touches the given curve, the law of increasing cost prevails according to the second definition, the law of diminishing cost according to the first definition; after the limit Q, the law of diminishing costs in both senses.

Now as to the sense in which Professor Seligman uses the terms, the first definition is suggested by the following passage :—

" If . . . an industry obeys the law of increasing returns or diminishing cost—where each increment in the amount produced costs less than the last " (*Shifting and Incidence*, Second Edition, p. 205).

But the context shows that the second, not the first, definition is in his mind : —

" If he produces less, each unit will, on the supposition that he has been producing under conditions of increasing returns cost him, exclusive of the tax, more than before." (*Ibid.*)

Similarly, the statement that, under the condition of diminishing returns, " each additional increment of production costs more than the last," is explained away by the context. The author's diagrams (*op. cit.*, pp. 209–210) leave no doubt as to his use of the terms. It is the cost per unit which he takes as increasing or diminishing with the law of increasing or diminishing cost. Compare his frequent use of the phrase " ratio of product to cost " (pp. 192, 211, 273, 278).

I do not complain of his employing the terms in a sense which is both useful and usual. All that I am concerned to maintain is (*a*) that my proposition in my sense of the terms is true, and (*b*), that his proposition in his sense of the terms is not so.*

(*a*) We may follow Professor Seligman in first supposing the law of constant cost to prevail, and afterwards substituting the law of increasing and diminishing returns respectively, other things being (as much as possible) unchanged. Only, with reference to definition I, " constant cost " must be interpreted to mean, not that the cost per unit is constant, but that the increment of the total cost per increment of product is constant; in other words, that the total cost curve is a right line, but not necessarily a right line passing through the origin, as definition II requires.

In the annexed, as in the former diagram, let the axis X represent total produce, and let the total cost curve, illustrated by the former diagram, pass through Q. Let the curve Ss represent, by its ordinate, the gross receipts (product $\times$ price) corresponding to any value of the product X. The position of maximum advantage to the monopolist is where the difference between the gross receipts and the total cost is a maximum: that is, at a point where the tangent to the cost curve is parallel to the tangent at the corresponding point of the gross receipts curve.[1] Thus, if the total cost curve is a right line BB' passing through Q (vertically above P) in order that OP should be the amount supplied, the line must be parallel to the tangent at S (vertically above Q and P) to the curve Ss.

* These words may still stand, although the proposition, occurring third in the edition of *Shifting and Incidence* which is now to be disputed, is not the same as that which, occurring in the *second* edition of that work, was disputed in the original article.

[1] Compare the illustration given III. 92, noticing that the curve BB' there represents net profits, not as Ss, here gross receipts. In order that there should be a maximum (the law of constant cost prevailing) the curve Ss must be *concave* towards the axis OX.

Now let us introduce successively the incidents of increasing and of diminishing cost; neither altering the law of demand, nor the amount of the total cost nor that of the cost per unit at the point P, and accordingly neither altering the price, nor the amount supplied, OP.[1]

It is evident that these conditions can only be fulfilled by a curve (AA') convex to the axis of X, and a curve (CC') concave to it (corresponding respectively to increasing and diminishing cost in my sense of the terms), each curve touching the line (BB') at the point Q.

Now let a tax of so much per unit be imposed on the mono-

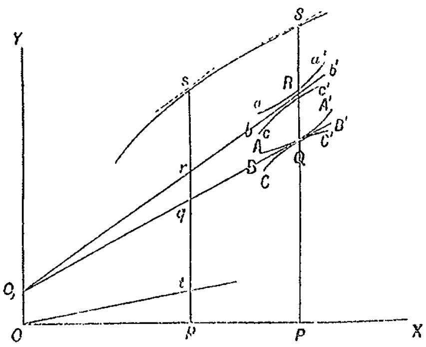

FIG. 2.

polised article. The effect of the tax is to push up every point on the total cost curve to an extent which is determined by the following construction :—[2]

Through O draw the right line Ot, making an angle with the axis of X such that at any point on the line, t, the perpendicular tp may represent the (total) tax paid on the product Op.[3] To find the displacement of any point, q, on the cost curve, draw a vertical line through q and measure upwards qr equal to pt intercepted between the lines OP and Ot. This construction holds for the curves of varying as well as for that of constant cost. Accordingly the new laws of cost formed by superadding the tax to the old cost will be related as shown in the diagram. The representation of constant cost will still be a right line (bb'), only inclined at a

[1] See the remarks on p. 159, par. 3, below.

[2] Compare Messrs. Auspitz and Lieben's construction for the representation of a specific tax.

[3] The tax tp = the product Op × the tangent of the angle pOt.

greater angle to the axis of X than the old line (BB'). The law of increasing cost will still be represented by a curve (aa') convex to the axis of X, the new curve touching the new right line at R. The law of diminishing cost (cc') will similarly retain, after the increase of cost by the last, both the character of concavity and the incident of contact with the right line representing constant cost.

Let us now compare the additions to the price consequent on the change of cost in each of the three cases. In the case of constant returns we have by construction the slope of bb' greater than that of BB', the slope of BB' equal to that of the tangent at the point S to the curve Ss, the slope of the tangent to this curve increasing as we move towards O, and diminishing as we move from it.[1] Therefore, to find the point at which the slope of the line bb' is the same as the slope of the tangent to the curve Ss at the corresponding point, we must move towards O, diminishing OP, say to Op, at which point the monopolist's profit is a maximum for the new law of cost. In the case of increasing cost the initial slope at R is the same as that of the line bb'. Therefore, by a parity of reasoning, we must move to the left in order to reach a point at which the slope of the cost curve may be the same as that of the gross receipts curve. But as we move to the left, whereas the slope of the right line remains constant, the slope of the convex cost curve diminishes. Accordingly the point at which the slope of the cost curve becomes equal to the slope of the gross receipts curve will be sooner reached in the case of the convex cost curve than in the case of the right line representing constant cost. By a parity of reasoning the maximum point will be later reached in the case of the concave cost curve. That is to say, the diminution of the supply will be less, and therefore the addition to the price less for the law of diminishing returns, corresponding to the convex curve, than what it is for the law of increasing returns, corresponding to the concave curve. *Quod erat demonstrandum.*

(*b*) Now let us consider Professor Seligman's proposition in his sense of the terms. To show the irrelevance of the distinction between increasing and diminishing returns according to the definition adopted by our author, suppose that, other things remaining constant, the total cost is varied by taking the origin now at a point above O_1, now at a point below it, *e. g.* O. The more or less of rise in price still goes with the concavity or convexity of the cost curve, that is, with increasing or diminishing returns in my sense; the rise is neither more nor less whether the

[1] See note to p. 154.

new origin is above O_1 or below it, whether it is a case of diminishing or of increasing returns in Professor Seligman's sense. My proposition remains true in whatever light contemplated; his proposition changes its quality with the accident of the theorist's point of view.

It would however be inaccurate to describe the law of increasing (or diminishing) returns in the second, the author's, sense as having no relation to the greater or less pressure of a tax. For there is a certain affinity between increasing (or diminishing) returns in the two senses. If the law of increasing (or diminishing) returns is fulfilled in the first sense continuously from the zero of production onwards, then it will be fulfilled in the second sense also.[1] Even if the curve of total cost has not been concave or convex *ab initio*, yet if it afterwards becomes so and remains so continuously, or at least for long tracts, then it is likely that many parts of the curve which fulfil one of the laws in the first sense will fulfil it also in the second sense. The very fact that the terms have been used by eminent economists in both senses without distinction forms a presumption that there is some affinity. In virtue of this affinity there is some connection between each of the laws in our author's sense and the property which he connects with it. There is a certain correlation; but not the coexistence which this statement implies.*

(3) The next question is, whether the tax of a monopolised article is more or less likely to hurt the consumer, according as the elasticity of demand is greater.** The subject is thus introduced by Professor Seligman; apparently without distinction in this first statement between competition and monopoly.

"If, thirdly, the demand is elastic as in the case of minor luxuries and of all comforts, that is of the general mass of commodities . . . the tax will be divided between the consumer and the producer. The proportions in which this division will take place will depend, so far as this element is concerned, chiefly on the elasticity of the demand. The more persistent the demand, the greater is the proportion of the tax which the producer will be able to add to the price; the more sensitive the demand, the smaller the sum by which he will find it profitable to increase the

[1] *Cp.* above, p. 70.

* This sentence has been substituted for the more damnatory verdict referring to the Second Edition.

** I use the term "elasticity" in this context in the same sense as the author whom I am criticising; that is, as I have pointed out (ζ, II. 394), not the received exact sense, but accurate enough for most of the reasoning here. Where any difference would be caused by the use of the proper definition I employ "elasticity" in inverted commas to denote the popular, Professor Seligman's, conception.

price. In other words, the greater the elasticity of the demand, the more favourable—other things being equal—will be the situation of the consumer " (p. 191).*

In a later passage, connected with the above by a footnote, it is written :—

" If the demand falls off greatly with every increase of price—or, in other words, if the margin of effective demand is small—the price, as we have seen, will be increased by much less than the amount of the tax, and the producer will suffer most of the loss. Conversely, if the demand is not so elastic—if an increase of price will produce only a small decrease of demand—a larger proportion of the tax will be added, and the consumer will suffer more than the producer " (p. 204).

In this passage the author is referring specially to " a monopolised industry," subject to the law of constant returns. He goes on to consider the cases of increasing and diminishing returns, both in monopoly and competition, concluding—

" In all these cases—whether of competition or of monopoly, of increasing or of diminishing cost—the important point remains, as before, the elasticity of demand " (p. 208).

The passages, as I interpret, contain rather a statement than a demonstration of the author's theory respecting the influence of greater or less elasticity. For what is called " a formal proof " we are referred, in the last but one of the passages above quoted, to the following passage :—

" It was stated above " [referring to the passage at p. 204, which I have already transcribed] " that the more elastic the demand, the smaller the proportion of the tax that he (the monopolist) would be apt to shift to the consumer. It may be wise to illustrate this also by some simple arithmetical figures " (p. 277).

As these figures are from an example given on the immediately preceding page, it may be well, parenthetically, to explain that in that example there was supposed a law of demand such that there was demanded :—

At price	\$5	....	1000	units of commodity.	
,,	$5\frac{1}{4}$	....	900	,,	,,
,,	$5\frac{1}{2}$	....	825	,,	,,
,,	$5\frac{3}{4}$	....	750	,,	,,
,,	6	....	700	,,	,,

The cost is supposed to be constant, viz. \$2 per unit.

* The reference in this and following passages is to the *second* edition of *Shifting and Incidence*. There seems no objection to retaining the references as given in my original article relating to theories which the author still maintains, as intimated at page 248 of his *third* edition. I advert to the third edition in a later article designated ξ in the present Collection.

The author continues :—

" Demand is said to be more elastic when each successive increase of price leads to a greater falling off of demand. The example above was based on the assumption that, at the price of $6, the demand would fall to 700. Let us now assume that, with a more elastic demand, the sales at price $6 would fall off as far as 675 units; and let us further assume that, with a more stable demand, the sales at the price $6 would fall off only to 725 units. Now, with the more elastic demand, the net profits would be, after the tax of $1 per unit was imposed, (6—3) × 675 = $2025; but with the less elastic or more stable demand the net profits would be (6—3) $725 = $2175. Hence the more stable the demand the greater the chances of his increasing the price by the whole tax."

[The author had shown on a preceding page that if a tax of $1 per unit is imposed the price will be raised by $1 " the entire tax will be shifted to the consumer."]

Here, as in the case of the former issue, one must admire the true logical instinct which leads the inquirer to introduce the attribute under consideration, increase of elasticity, other things being preserved the same, in order to observe the effect due to the introduced attribute. But, as writers on logic have pointed out, the *method of difference* is not wholly empirical : it often requires a good deal of *a priori* knowledge obtained by deduction in order to divine not only what " other things " may be allowed to vary as being immaterial to the matter in hand, but also what other things ought *not* to be kept the same. Suppose it to be inquired whether youths at a certain age, say between thirteen and fifteen, increased in weight with a particular degree of rapidity. In comparing the weights of youths at the two ages, it would be proper to keep some things the same—the persons themselves, for example. But it would not be proper to keep constant the *heights* of the youths under observation : it would not be proper to select instances in which there was no growth in height, in order to test whether in general there is a considerable growth in weight. For it is knowable *a priori* that there exists *correlation* between height and weight. The specimens selected for their constancy in height are not fair specimens of the change in weight. Now Professor Seligman selects his specimen with an analogous partiality. In order to test the effect of increased elasticity on taxation he had to construct a new law of demand, more elastic than the original one. He is within his rights in supposing, as he does in effect, that the new demand curve passes through the point which corresponds to the old price, the price before the change of elasticity and before the tax. But he is not within his right when he in effect takes as a type on which to build a general argument a law of demand equivalent to a curve which not only passes through the point corresponding to the old price, but also is coincident with the old demand curve, for a considerable tract of price, for nearly the whole extent by which the price (before the change of elasticity) was pushed up by the tax. For, in general, the new demand curve ought to be conceived as *cutting* the old one at the point specified. Whence it is deducible that in general, when the law of elasticity is thus raised, there will be some change of price *before the tax*. This preliminary change is the attribute which corresponds to *height* in my apologue. The author has selected an instance which may be seen *a priori* to be favourable to his conclusion. An instance selected impartially at random, so to speak, would be very unlikely to present the phenomenon of both price and quantity unchanged by the change of elasticity. In fact did it indeed ever occur to any one, wishing to illustrate what Mill calls the increase of the demand in a greater or a less proportion than the cheapness,[1] to conceive a variant so peculiar as that which forms Professor Seligman's illustration?

The following is a more typical example. Let the law of demand be originally such that there is demanded :—

At price, $5	....	1000 units of commodity.	
,, $5\frac{1}{4}$	....	900 ,,	,,
,, $5\frac{1}{2}$	....	800 ,,	,,
,, $5\frac{3}{4}$	....	650 ,,	,,
,, 6	....	475 ,,	,,

the cost being, as before, constant, $2 per unit of commodity. Then, as in Professor Seligman's example, originally the monopolist will fix the price at $5. The net profit at that price, as shown in the accompanying statement, will be higher than at any other of the prices.

BEFORE THE CHANGE OF ELASTICITY.

	Output	1000	900	800	650	475
	Price	5	5·25	5·5	5·75	6
Before the tax.	Price *minus* cost ...	3	3·25	3·5	3·75	4
	Net profits	3000	2925	2800	2437·5	1900
After the tax.	Price *minus* cost ...	0·3	0·55	0·8	1·05	1·3
	Net profits	300	495	640	682·5	617

[1] *Political Economy*, III. ch. xviii. § 5.

AFTER THE CHANGE OF ELASTICITY.

	Output	1000	900	800	650	475
	Price	5	5,775	6·05	6·325	6·6
Before the tax.	Price *minus* cost	3	3·775	4·05	4·325	4·6
	Net profits	3000	3397·5	3240	2811·25	2185
After the tax.	Price *minus* cost	0·3	1·075	1·35	1·625	1·9
	Net profits	300	967·5	1080	1056·2	902·5

Now let a tax of $2·7 per unit be imposed. The resulting figures given in the accompanying statement show, as far as a discontinuous schedule of this sort can show, that the price will now be raised from 5 to 5¾, that is by $¾ in consequence of the tax. Next let us introduce the circumstance of diminished elasticity. Understanding that after, as before the change of elasticity, 1000 units are demanded at the price 5, let us suppose the demand to be tilted up for each higher price in that neighbourhood. For example, whereas originally an output of 900 was carried off by a price of 5¼, let that price now become 5¼ × 1·1, and let the other prices be increased in the same proportion. Then we shall have, after the change of elasticity, the prices specified in the second part of the annexed statement, corresponding to the original outputs. That is before the tax by the mere fact of diminished elasticity the price has been raised from 5 to 5·775. Now superadd the tax of 2·7 per unit, as before, and it will appear that the rise of price is from $5·775 to $6·05, that is, $·275, considerably *less* than the rise of price under the condition of greater elasticity, which was ·75.

But of course this manipulation of figures is " mere palpation," affording no certain warranty of general truth, and rather calculated to obscure essential points, as in the memorable instance of an arithmetical example constructed by J. S. Mill and condemned by Professor Marshall.[1] A hundred, perhaps a thousand, empirical instances, taken impartially at random, might be required in order to elicit by a laborious elimination of chance the tendencies which may be discovered at a glance upon inspection of a few symbols.

(4) The next question is whether in general, or with what degree of generality, when a specific tax is imposed on a monopolised article, the price will rise. Professor Seligman now admits " that in ordinary cases the monopolist will shift at least a part of the burden." But he ventures " still to cling to the position " that " cases may arise in which it will be profitable for the monopolist to bear the burden himself. No part of the tax will be shifted to the consumer." He disputes my position that the tax

[1] *Principles of Economics*, Book VI. ch. ix., note on Ricardo's doctrine.

will affect the consumer, except " in two special cases, (1) where it is not in the power of the monopolist to increase or limit his output at will; (2) where the monopolist is a sole *buyer*, and the supply of the article bought is perfectly inelastic; for instance, a combination of tenants dealing with landlords [1] incapable of combining." [2]

" That these are not the only cases, however," remarks Professor Seligman, " is clear from the argument in the text." Here is this argument :—It is supposed, as in the example quoted on our page, that the monopolist will sell 1000 units at price \$5, 900 units at price \$5¼, and so on as stated in the third column of the annexed schedule. The cost per unit is supposed to be constant, viz. \$2. Professor Seligman argues :

" His net profits then after a tax of ¼ of a dollar had been imposed, would be,"

At price	5	...	(5 − 2¼) × 1000	...	= \$2750.
,,	5¼	...	(5¼ − 2¼) × 900	...	= \$2700.
,,	5½	...	(5½ − 2¼) × 825	...	= \$2681·25.
,,	5¾	...	(5¾ − 2¼) × 750	...	= \$2625.
,,	6	...	(6 − 2¼) × 700	...	= \$2625.

" In other words the monopolist will continue to find his greatest profits in continuing to charge the original price."

He will, I rejoin, if he can only alter the price *per saltum*, by leaps of ¼ dollar. But surely it was not necessary in an article on *pure theory* to notice this obvious limitation, which may fairly be relegated to the category of *friction*. If the monopolist can adopt an intermediate price between \$5 and \$5¼, I submit that he will tend theoretically to do so, for the reasons which I have given in one of the papers referred to.[3] Actually no doubt he may not do so, because the gain directly resulting from the change may not compensate the incidental disadvantages attending a change. This force of friction is well described by Professor Seligman, and is all the more clearly discerned by the mathematical economist, in that he perceives, as pointed out by Professor Knut Wicksell,[4] that when the tax is small the gain must be *very* small, of the second order.[5]

[1] *Pure Theory of Taxation*, S, II. 91.

[2] Assuming, of course, that the landlords have nothing else to do with their land but to offer it to tenants. Professor Seligman seems not to accept this postulate. He says, " the landowner is not compelled to part with his land; but the tenant is compelled to occupy some apartments " (*op. cit.*, p. 242).

[3] See especially the diagrammatic statement in the review of Professor Graziani's theory (ζ, II. 399, and III. 92).

[4] In the admirable study of the subject in his *Finanztheoretische Untersuchungen*, which I had not seen when writing before, in the ECONOMIC JOURNAL, 1897, on this topic.

[5] In the symbols which we have employed above, the gain is of the order

I don't know that there remains anything worth fighting for under this head. I quite admit—I never denied—the efficacy of friction. Professor Seligman appears to admit the abstract theory when, in a passage which has been already quoted, he reasons : " the producer, who has advanced the tax, will increase his price only to that point where the smaller sales are compensated by the higher price, so that his net profits will still be at the maximum " (p. 204). If any difference of opinion remains, I surmise that it relates to the assumed continuity of the demand-curve (and other economic functions).[1] I have thought it legitimate to assume, not only with Professor Marshall, that "the demand for a thing is a continuous function," [2] but also that, like the continuous functions which we ordinarily meet with in nature, it is not continually changing its character in respect of convexity or concavity.* If the gross receipts curve represented by *Ss* on our diagram 2 is concave to the axis of X at the point S corresponding to maximum net profits, it may be assumed that, in general, in the great majority of cases which occur in ordinary practice, the curve will retain that character, as we move away from the point, for some finite distance. On this ground, it may be assumed as generally true that the imposition of a tax will tend to raise price.

In the postulate of continuity lies the answer to the difficulty raised by Professor Seligman in the following passage :—

" Cournot states that the tax must always be shifted (except in the cases mentioned in the preceding note.") [The cases quoted from the article in the ECONOMIC JOURNAL, 1897 (S, II. 91). Does Cournot make both these exceptions ?] " Professor Edgeworth (ECONOMIC JOURNAL, VII. p. 405) thinks that this is true ' in general.' Later, when hard pressed by Professor Graziani, he seeks to maintain his position by assuming ' that the change of price is small,' ' by taking Δp *sufficiently small* ' (ECONOMIC JOURNAL, VIII. p. 235). But is it fair to assume that a small change of price is ' more general ' than a great one ? And would Professor Edgeworth's elaborate formulæ all hold good, if the change of price were substantial ? " (p. 276).

Certainly, the formulæ hold good for substantial changes of

$\frac{1}{2}\tau\Delta x$; upon which, however, it may be remarked (1) that Δx, though in general of the same order, may be sometimes much larger than τx, (2) that $\tau\Delta x$ though small in relation to the *gross receipts* may be considerable in relation to the *net profits*.

[1] *Cp.* Professor Seligman *op. cit.* p. 278 (note) : " The error of Professor Edgeworth seems to consist in the assumption that the demand curve is continuous."

[2] Preface to *Principles of Economics*.

* See below, ζ, II. 389.

price as long as the conditions of a maximum continue to be fulfilled, that is, presumably for some finite distance.* On the page following that which Professor Seligman has quoted, there is given an illustration, in which, as the tax is increased, the price will continue to rise up to the point where the monopolist's profits vanish. The rise of price attending increase of taxation may be interrupted when the demand curve (or some other function involved) changes its character in respect of concavity. Professor Seligman's own example affords a good instance. Taking the figures quoted above, with the omission of the tax, we have the subjoined data for the gross receipts curve. It will be seen

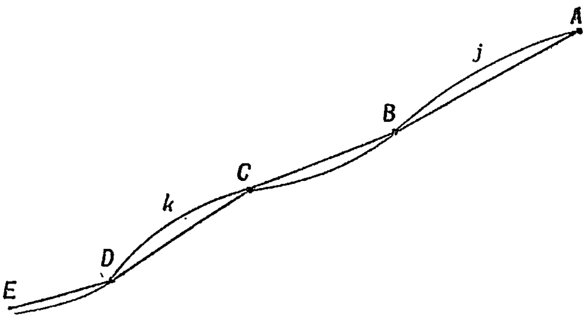

FIG. 3.

	A		B		C		D		E
Output	1000		900		825		750		700
Price	5		5·25		5·5		5·75		6
Gross receipts	5000		4725		4537·5		4312·5		4200
Differences of ordinates ...		275		187·5		225·0		112·5	
Differences of co-ordinates		100		75		75		50	
Slope		2·75		2·5		3		2·25	

that as the price is lowered from 6 labelled E to 5 labelled A, the output of course being concomitantly increased, the gross receipts continually increase. But the rate of this increase is not continuous. It is less rapid from *E* to *D* than from *D* to *C*, and from *C* to *B* than from *B* to *A*. The simplest continuous curve corresponding to the data would presumably be of the form indicated by the sinuous line in the annexed diagram. The reader will be so good as to substitute in imagination a curve of this kind, instead of the gross-receipts curve *Ss*, in our diagram 2, on a former page. By the reasoning there given, or referred to, it

* *Cp.* below, p. 167.

appears that, with the imposition, and gradual increase from zero, of a specific tax, the output will decrease, and the price will increase, as long as the curve is concave, say up to the point j. At that point the sort of index which we may conceive moving along the curve, stops. It may stop some time at j; or it may almost immediately fly over to the next crest, between D and C. It cannot descend to C (the law of cost being supposed constant, $2 per unit) unless the curve is very unusually complicated. It will continue to move on to the point k, where the curve again changes its curvature, after which another jump may sooner or later ensue.

Because the mathematical investigation advances by tentative steps it is not precluded from going in the direction of the rise of price as far as any other method, provided that the conditions of a maximum are secured. Without that condition, the calculus is helpless: it "fears to tread" where the ground is insecure; contrasted in that respect with other methods, but not to its disadvantage.

(5) Professor Seligman draws out his arrays of figures for another pitched battle on the question whether "a tax on monopoly gross receipts *must* raise prices"; maintaining that, "although it is generally true that a tax on monopoly gross receipts will raise prices, this conclusion not necessarily follows." From the mathematical point of view the distinction between this case and the preceding is unimportant, with respect to the purpose in hand; as appears from Cournot's analysis.[1] Suppose with Professor Seligman that a tax of 10 per cent. is imposed on gross receipts, then the amount which the monopolist seeks to maximise is $\frac{9}{10}$ gross receipts — total cost; or, $\frac{9}{10}$ (gross receipts — $\frac{10}{9}$ total cost). Accordingly, the change of price consequent on the tax will be the same as if, instead of the tax *ad valorem*, there had been an increase of the total cost by 11·1 per cent. The effect of such an increase of cost on price is identical with that of a specific tax in the case of constant returns, and of the same general character in the case of varying cost: as may be seen from our diagram 2, by observing that any point on the cost curve is pushed upwards as before, not now to the extent of a certain proportion of the abscissa, but to the extent of $\frac{1}{9}$th of the ordinate. The same theoretical necessity, the same practical reservations, apply to the fifth as to the fourth issue.

Let us pause after these five rounds. I have noticed some half-a-dozen other instances in which my distinguished opponent's

[1] *Principes Mathématiques*, Art. 41, ch. vi.

conclusions are diametrically opposed to those which are deducible by mathematical reasoning. But I think it best to confine the present discussion to issues which have been already raised; respecting which a difference of conclusion may fairly be ascribed to the difference of method rather than a mere slip on either side.

II. I go on therefore to consider some general reflections on the mathematical method which the author has prefaced to the discussion of particular theorems.[1]

Here is one of these reflections.

(1) " The mathematical study of the pure theory often assumes a simplicity of condition which does not actually exist; it purposely neglects the all-important element of friction and constructs hypotheses irrespective of their agreement with the facts of actual life " (p. 173).

I quite admit that mathematical reasoning—like all abstract reasoning—has its abuses, as well as its uses. I only enter a *caveat* against its being supposed that this remark is particularly relevant to the present discussion. The abstract questions which are at issue are understood in the same sense by both parties; there is no reason to suppose any difference of opinion as to the value of the right conclusions. What the impartial spectator has to consider is whether the party that dispenses with mathematical reasoning obtains the true answers. To cry out *Ne sutor super crepidam* does not prove that it is possible to make good shoes without the proper tools.

Again Professor Seligman remarks :

(2) " The chief advantage of the mathematical method is seen in the use of diagrams where an intricate point which involves the simultaneous consideration of several causes can be illustrated with greater brevity and clearness than in any other way. But when we proceed from diagrams to the higher algebra, the use of the mathematical method sometimes leads to refined calculations of more importance to the mathematician than to the economist, and of little perceptible use in solving any practical economic problems " (p. 173).

This unqualified preference of diagram to symbol appears to me to be exaggerated. When the causes to be simultaneously considered are numerous, diagrams are apt to become helpless. Thus in order to treat our first question diagrammatically, it would have been necessary to resort to *solid* geometry. And, as there are only three dimensions of space, even solid geometry

[1] *Op cit.* pp. 172–3.

would be inadequate to illustrate the problem of *three* classes of fares. Even with respect to the other issues, where we are concerned with only one commodity, the use of symbols appears to me to have some advantage. I propose to illustrate this statement by restating in symbolic language, specially addressed to the mathematical reader, solutions of the four problems—(2) to (5) inclusive—which have already been treated otherwise.

Let us begin with Cournot,[1] by considering an indefinitely small tax or addition to taxation, i in his notation, or, as we might say, $\Delta\tau$. It follows from Cournot's reasoning that the increment of the price consequent on an increment of taxation may be expressed in terms of the following quantities: (1) the price, say p; (2) the rate at which the total cost increases with the product, say c; (3) the rate at which the increase of the total cost attending an increment of product increases with the increase of product, say c'; (4) the "elasticity," * or rate at which the amount demanded diminishes as the price is increased, say e; and (5) the rate at which the "elasticity" increases with the increase of price, say e'. Substituting these symbols in Cournot's expression for the change of price consequent on a tax (in his Art. 38, or rather in the expression which he gives for the change of price consequent on an increase in the cost of production equivalent to a specific tax in his equation (4), Art. 31) we have, *mutatis mutandis* [2]

$$\{-2e - c'e^2 - e(p - c)\}\Delta p = -e\Delta\tau.$$

Whence $$\frac{\Delta p}{\Delta\tau} = \frac{e}{2e + c'e^2 + e'(p - c)}.$$

To apply now this formula to the problems in hand. We see at once that to a (positive) increment of taxation corresponds an increase of price. This proposition holds good alike for specific and *ad valorem* taxes—our (4) and (5) (compare Cournot, Art. 41). And what is true of an indefinitely small increase of taxation, is true of a finite increase, so long as the denominator in the expression for $\frac{\Delta p}{\Delta\tau}$ continues positive; that is, I think we may say "in general"—in the long run of cases—to some finite distance from the point at which we started, as shown by Cournot, Art. 32.

[1] *Principes Mathématiques*, ch. vi. Art. 38.

* As announced above (p. 157), inverted commas are applied to elasticity in the popular sense corresponding to Cournot's $F'(p)$, when it becomes important to distinguish that sense from the strict definition corresponding to Cournot's $F'(p)p/F(p)$. The distinction becomes significant when the *second* differential coefficient makes its appearance (see ζ, p. 394).

[2] It will be observed that his $F'(p)$ is identical with our $-e$, his $\psi'(p)$ with our $-c\,e$, his $\phi'(D)$ with our c, his $\phi''(D)$ with our c', his $F''(p)$ with our $-e'$.

As he says, "this method of demonstration should be borne in mind, as it will be frequently recurred to." Bearing it in mind with respect to the remaining problems above designated, (2) and (3), we need only examine how $\frac{\Delta p}{\Delta \tau}$ is affected by the incidents in question, namely, variations in the law of cost, and variations in the "elasticity."

Considering the formula above given, we see that both the numerator and the denominator[1] of the expression for $\frac{\Delta p}{\Delta \tau}$ being essentially positive, $\frac{\Delta p}{\Delta \tau}$ must decrease with the increase (and increase with the decrease) of c', other things being the same. The only other relevant things are e, e', p and c. And the only significant question is whether we have any reason to think that any of these quantities is likely to be greater or smaller when c' is greater, in general in the long run of cases. I submit we have no ground for thinking that there is any *correlation* between c' and any of those variables. Accordingly in the long run the rise of price consequent on any assigned increase of taxation is likely to be greater the smaller c' is. A particular case of this proposition is that the rise of price is likely to be greater when c' is negative than when c' is positive: in other words, higher when the law of increasing, than when the law of diminishing, returns prevails. That is, understanding those laws as I have defined them. There is no direct connexion between increasing and diminishing returns in the other sense. But the proposition which has been enunciated is true also in the second sense so far as the attribute, which forms the first definition, is apt to be attended with the attribute which forms the second definition[2]—that is, possibly, very far.

We come lastly to problem (3). How is $\frac{\Delta p}{\Delta \tau}$ affected by the increase or decrease of e? It is quite a relief, after the monotony of contradiction, to have to admit that I have committed a slip at this point. In my former version of the theory[3] in the expression for "the increase of price due to a small tax" corresponding to the expression for Δp just now written, I put, for the sake of simplicity, a single symbol, B, for what I now call $-c'e^2$. And that was all right for the immediate purpose in hand. But in

[1] The negative of this denominator being, as pointed out by Cournot, Art 31, "necessarily negative, according to the well-known theory of maxima and minima."

[2] *Cp.* above, p. 70.

[3] ECONOMIC JOURNAL, Vol. VII., p. 227, note 2. [Omitted from S.]

applying the formula to enunciate the effect of a change in e, I treated B as constant,[1] forgetting that it involved e.

The following is, I now think, a more correct statement. In any given case it is impossible to say whether the increase of elasticity conduces to the increase or the decrease of the efficacy of a tax to raise price; unless we are given not only c' (which may be supposed), but also e', involving the *curvature* of the demand curve, which is not, I think, usually given,[2] even as to sign, much less with the quantitative precision which would often be necessary for the present purpose.

But the expression for $\frac{\Delta p}{\Delta \tau}$, though perfectly indeterminate for any particular case, may afford, I think, a certain presumption for the long run. Suppose the sign of c' to be given, *e.g.*+, the law of decreasing returns prevailing. Then in the long run of cases—while e' is now positive, now negative in sign, now large, now small in absolute quantity—for a majority of those cases an increase of e would be attended with an increase in the denominator of the above-written expression for $\frac{\Delta p}{\Delta \tau}$, or rather what it becomes when both numerator and denominator are divided by e, and, therefore, a decrease of the ratio under consideration. Conversely, when the law of increasing returns prevails, c' being negative, an increase of elasticity is likely to be attended with an increase in the efficacy of taxation to raise price.

To the extent of the former clause I have to retract my original statement. But I am still able to affirm universally, without reference to the law of cost, the contradictory of Professor Seligman's theory that "the greater the elasticity of demand the more favourable—other things being equal—will be the position of the consumer"; if the situation of the consumer is tested, as it ought to be, not so much by the rise of price as by the loss of *consumers' surplus*. Employing the proper criterion of the consumers' welfare, we may affirm, "the greater the elasticity of demand the more *un*favourable—other things being equal—will be the situation of the consumer."[3]

[1] ECONOMIC JOURNAL, Vol. VII., p. 228, note 4 continued from p. 227.

[2] As to this unknown element, see E, I., p. 135, and ζ, II. 394.

[3] The *loss* of *consumers' surplus* consequent on raising the price from p to $p + \Delta p$ is approximately (*cp.* E, pp. 117, 132) $x\Delta p$; where x is the product, which, by Cournot's equation (3) of ch. v. $= e(p-c)$. Therefore the loss of *consumers' surplus*

$$= e(p-c)\Delta p = \frac{(p-c)e^2\Delta\tau}{2e+c'e^2+e'(p-c)} = \frac{(p-c)\Delta\tau}{\frac{2}{e}+c'+e'\frac{(p-c)}{e^2}}.$$

This expression for the *loss* of consumers' surplus is always *positive*, both the

For the purpose of obtaining propositions in Probabilities, as the preceding may be described, I submit that symbols seem to have an advantage even over diagrams—not to speak of numerical illustrations. Thus it may be objected to our diagrammatic proof [1] of proposition (2) that some *a priori* knowledge derived from analysis is required to guarantee the legitimacy of our supposition that the law of cost only is varied, other things being preserved constant. A diagrammatic proof of proposition (3) would be even more precarious.

(3) One more of Professor Seligman's general reflections :—

" It may even be doubted whether the mathematical method has independently discovered any important principle susceptible of practical application that could not have been also expressed in everyday language."

Those who have followed the preceding discussion may be disposed to admit that, if the mathematical method does not itself discover important practical principles, it may at least be usefully employed to test the principles which a distinguished practical economist regards as important. If it is worth his while to employ some pages of economic analysis and numerical examples in endeavouring to prove those principles, it is worth our while to employ some lines of symbol in endeavouring to disprove them.* The negations are often also affirmations, but not very confident ones. It is as if an opponent should prophesy that the last week of April or May would be the coldest part of the month. The reply is that what we know about the matter points in a contrary direction: there is a constant cause making for greater heat—namely, the position of the earth relatively to the sun—in the latter part of each month; though doubtless that tendency may be counteracted by unpredictable vicissitudes of weather. What if the more abstract part of political economy, like the more sublime part of astronomy—that which contemplates the mechanism

numerator ($= \Delta\tau x \div e$) and the denominator (for the reason given in note 1 to p. 168) being positive. The loss is in the long run greater the greater e is, since one term of the denominator $\frac{2}{e}$ becomes less as e becomes greater; while the other term of the denominator which involves e may be treated as inoperative (if not diminishing), on an average, in the long run of all possible values (positive and negative) of e', in our ignorance of e'. Thus the loss of consumers' surplus is likely to be greater the greater the elasticity.

[1] Above, p. 155.

* There are here omitted some sentences referring to the *second* edition of *Shifting and Incidence*, but less pertinent to the *third* edition.

of the heavenly bodies external to our system—were not at present susceptible of direct practical application, the mathematical theory of economics might still confer a benefit analogous to that which the mathematical theory of astronomy conferred when it discredited the pernicious pretensions of the astrologers. There are those who think that even of the received economic analysis the most important function is *negative*. Thus Mr. Leslie Stephen :—

"Political economy, as I venture to think, has been especially valuable in what I have called its negative aspect. It has been more efficient in dispersing sophistries than in constructing permanent theories. Economic writers have exploded many absurd systems. They have so far cleared the way for an application of sounder methods. But the complexity of the problem is so great. . . ."[1]

The sort of sophistry which has been eradicated from the general field of economics by the received *organon* finds a still virgin soil in the nooks and corners of which the cultivation requires the implements of mathematics.

The trenchancy of this criticism is not inconsistent with the diffidence which is proper to an inexact science, and the respect which is due to a high authority. For on the one hand, the region of hypothetically abstract theory, to which this polemic is confined, forms the one territory of economics in which issues may be fought out without compromise, there being a right diametrically opposed to the wrong. And on the other hand, it is no discredit to the ablest combatant, when he is unprovided with the proper weapons, to succumb.

[1] *Life of Fawcett*, p. 149.