(I)

TESTS OF ACCURATE MEASUREMENT.

[This is the *second* of the Memoranda on Variations in the Value of Money, prepared for the British Association in the latter 'eighties of last century. A separate place is assigned to the second Memorandum, which purports to test the accuracy of the calculations prescribed in the first (and third). The verification is effected by the use of that leading principle of Probabilities known as the Law of Error. Caution is required in applying the test. The beginning of wisdom in this matter is to recognise the analogy between the grouping of heterogeneous price-variations and the dispersion of errors-of-observation. But it is not safe to treat each relative price as an *independent* observation. In the case of prices there is no doubt a good deal of independence in the sense proper to Probabilities. But there is also—as perhaps more than is commonly assumed in the case of physical observations—a good deal of interdependence or correlation among the factors which compose or cause the given observations. The subjoined computations are made in the supposition that each element of an index-number, each percentage representing a comparative price, is "subject to a presumably independent error" (par. 5). So far as this hypothesis does not hold good in the concrete, the inverse square root of n, the number of the data, which continually figures in the measure of possible inaccuracy must be taken *cum grano* (cp. *infra*, p. 324).

Attention may be called to the advocacy of the Median (for the computation of certain index-numbers) on the score not only of its peculiar facility, but also (in certain cases) its comparative accuracy. The Weighted Median, Laplace's Method of Situation, is not so familiar an operation but that its exemplification may be useful.

The reader may be assisted in following the computations by having before him an example to which they are often referred, namely, the model index-number proposed by the British Association Committee. It is reprinted here together with the explanations attached to it by Giffen, who took a principal part in the preparation of the scheme. This index-number and Giffen's explanation formed a part of the Second Report of the Committee which was drawn up by Giffen. The Memorandum by the present writer which is now reprinted originally appeared as an annex to that Second Report. An extract from that Report is appropriately included in these prefatory remarks.

Extract from the Second Report of the British Association Committee, drawn up by Giffen.

"The considerations we have to suggest as now most important practically, in preparation for more exact and complete measurements in the future, are the following :—

1. In the absence of retail prices—which it would be most convenient to use in forming a standard of *desiderata*—use must necessarily be made of wholesale prices only. No other prices are obtainable, and those prices must be preferred, in the selection of typical articles, where the records are best.

It appears, however, from the best consideration of the subject, that the differences likely to be made from the true result which would be obtained from a more complete record of prices are not likely to be material. On this head the Committee would refer to a paper by Mr. Edgeworth, which has been prepared for their use, and which is appended. The prices of articles taken without bias from a group are likely to be fairly representative of the average course of prices of that group.

2. While an index-number assigning relative weight to different articles so selected is an important means of arriving at a useful result, it cannot be said, in the present state of the data on the subject, to be an altogether indispensable means. The articles as to which records of prices are obtainable being themselves only a portion of the whole, nearly as good a final result may apparently be arrived at by a selection without bias, according to no better principle than accessibility of record, as by a careful attention to weighting. On this head the Committee may refer to the above paper of Mr. Edgeworth, which seems conclusive on the subject.

3. Practically the Committee would recommend the use of a weighted index-number of some kind, as, on the whole, commanding more confidence. But they feel bound to point out that the scientific evidence is in favour of the kind of index-number used by Professor Jevons—provided there is a large number of articles—as not insufficient for the purpose in hand.

Nothing is more remarkable in the comparisons of the recent index-numbers than the correspondence of the curves of general course of prices indicated. A *weighted* index-number, in one aspect, is almost an unnecessary precaution to secure accuracy, though, on the whole, the Committee recommend it.

4. The Committee have had before them a suggestion for a new index-number, which might be used for some official and private purposes, based on the practical considerations referred to, and making use of the best wholesale prices, while having regard to the ultimate standard of *desiderata*. The nature and object of this index-number are explained in the accompanying memorandum, which has the general approval of the Committee, though they do not consider it necessary here to go into all the details. The object is to provide something for which it would be possible to obtain and publish official prices, and by reference to which contracts could be made, and it is submitted for discussion and future reference.

5. It would be most desirable to supplement any such index-number by a good statistical account from time to time of the aggregate income of the people and the relative numbers and aggregates of incomes of different amounts. In some index-numbers in past times the wage of a day-labourer is inserted as one of the articles. This may have been correct enough for some purposes, and in the circumstances would not prevent the index-number from indicating the general changes in the value of money in the periods compared. But the more useful method would seem to be to distinguish between the human unit in production and the thing produced. Among the most important comparisons for which such figures are used at all are the effectiveness of labour at different times and places, and the command of the labourer or other earner over the amounts produced; and these comparisons can only be made when an independent standard of the production and consumption of the labourer is set up, with which his earnings may be compared. No argument is needed to show that, along with index-numbers as to prices of commodities, there should be an endeavour to ascertain the aggregate earnings of a community and the distribution of the earnings so as to show on the one side the command over commodities which different classes possess—the real as distinguished from the nominal incomes—and on the other side the relative effectiveness of the labour of a community at different times or of one community compared with another.

TABLE FOR THE CONSTRUCTION OF AN INDEX-NUMBER.

Statement showing the estimated amount of the expenditure on the undermentioned articles in the United Kingdom and the proportion of the amount in each case to the total expenditure on all such articles, with suggestions for an index-number based approximately on the proportions stated, but with modifications so as to substitute percentages in round figures; showing also the description of the specific wholesale article, the price of which it is proposed to use in the calculation of the index-number; giving also the price-list or other source from which quotations are to be obtained.

1	2	3	4	5	6	7
Heads of articles.	Articles consumed or used up.	Estimated expenditure per ann. on each article.	Percentage of each amount in column 3 to total.	Relative importance proposed for each article in index-number reduced to percentages.	Description of the specific article of which the price is to be quoted as typical.	Price-list or other source for price quotations.
Breadstuffs.		£ 000,000				
	Wheat	60	6·5	5	English wheat	Gazette average
	Barley[1]	30	3·25	5 } 20	,, barley	,, ,,
	Oats	50	5·4	5	,, oats	,, ,,
	Potatoes, rice, etc.	50	5·4	5	,, potatoes	Average import price
Meat and dairy food.	Meat	100	11	10	Mean of live meat per stone of 8 lbs. Smithfield	Weekly market quotations
	Fish	20	2·2	2½ } 20	Average per cwt. landed	Official returns (Board of Trade)
	Cheese, Butter, Milk	60	6·5	7½	Cheese	Average import price
					Butter	,, ,, ,,
Mass luxuries.	Sugar	30	3·3	2½	Refined sugar imported	Average import price
	Tea	20	2·2	2½	Tea imported	,, ,, ,,
	Beer	100	11	9 } 20	Beer exported	Average export price
	Spirits	40	4·3	2½	Spirits imported	Average import price
	Wine	10	1	1	Wine imported	,, ,, ,,
	Tobacco	10	1	2½	Tobacco imported	,, ,, ,,
Clothing.	Cotton	20	2·2	2½	Cotton imported	Average import price
	Wool	30	3·3	2½ } 10	Wool imported	,, ,, ,,
	Silk	20	2·2	2½	Raw silk imported	,, ,, ,,
	Leather	10	1·1	2½	Hides imported	,, ,, ,,
Metals and minerals.	Coal	100	11	10	Coal exported	Average export price
	Iron	50	5·4	5 } 20	Scotch pig-iron	Market price
	Copper	25	2·7	2½	Copper ore imported	Average import price
	Lead, zinc, tin, etc.	25	2·7	2½	Lead ore imported	,, ,, ,,
Miscellaneous.	Timber	30	3·3	3	Timber imported	Average import price
	Petroleum	5	·6	1	Petroleum imported	,, ,, ,,
	Indigo	5	·6	1	Indigo imported	,, ,, ,,
	Flax and linseed	10	1·1	3 } 10	Flax imported	,, ,, ,,
	Palm oil	5	·6	1	Palm oil imported	,, ,, ,,
	Caoutchouc	5	·6	1	Caoutchouc imported	,, ,, ,,
	Totals	920	100	100		

[1] There is a large consumption of barley, exclusive of its use in the manufacture of beer.

"In this table the first column indicates six leading genera which comprehend the twenty-seven classes of articles specified in the second column. These articles are either finished products (things ready for consumption, like cheese and milk) or *represent* such things by entering into their production, as coal (used in manufacturing) and timber, for instance, go to the production of houses and furniture.

"The third column gives in round numbers (000,000's being omitted) the average national expenditure on each class of article at present and for the last few years, and presumably also for the immediate future the *proportions* at least, if not the absolute amounts, of expenditure (such proportions, as shown in Mr. Giffen's reports on the variation in the prices of exports and imports, remaining pretty constant during a period of years). In the estimated amount of consumption allowance is made for the addition to the value made before the articles are in the form in which they are finally consumed.

"In column 4 these amounts (or proportions) are reduced to percentages (of the total amount expended on such articles).

"In column 5 the relative importance proposed to be assigned to each article in the index-number is stated, mainly on the basis of the percentages in column 4, but with modifications so as to substitute even figures for the convenience of handling.

"In column 6 the specific articles are described, of which it is proposed to obtain the prices as typical of the group really included on the corresponding line in column 2. Wheat, for instance, consists of many different kinds and qualities; the one quality and kind it is proposed to quote as typical of the whole is English wheat as returned officially to the Comptroller of the corn returns, which itself no doubt comprises many qualities. Of iron, again, there are innumerable qualities and kinds; it is proposed to take Scotch pig-iron, in which there are large dealings, as typical of the whole. The same with other articles. In most cases large groups are dealt with because the article selected is the average imported or exported, which includes many qualities, but it should be distinctly understood that in any case the most that can be done is to select specific articles which are typical of large groups.

"In column 7 the source from which the quotation of the specific articles mentioned in column 6 is to be obtained is stated.

"The above is of course only a rough suggestion for an index-number. Even if the method is generally approved of, many questions might be discussed as to the amounts of the annual consumption of each group of articles specified in column 2, as to the relative importance to be assigned practically in column 5, and as to the selection of the article in column 6 which is to be treated as typical of the group. It would be possible to introduce two or more quotations instead of one for a particular group if thought desirable, but this would be troublesome in working. For practical purposes there must not be too many articles. Mr. Edgeworth's mathematical deductions as to the consequences of taking the price of an article selected at random from a group, instead of the general average course of prices for the group, appear to justify the expediency of this procedure.

"Were such a general index-number introduced, and prices calculated upon it backwards and forwards, it would be easy to rearrange it for any special purpose, such as to give more or less weight to one or more groups according as they are assumed to enter into the consumption of a particular class of persons whose position at different times as affected by the course of prices is to be specially investigated. The index-number could also be compared with other index-numbers upon some other objective basis, such as the relative importance of each article in the import and export trade of a country; and index-numbers for one country and place could be compared with those for other countries or places. The index-number now suggested is only put forward as a convenient one, illustrating the variations in prices in England according to what is called the standard of *desiderata*, and which could be made use of—not neglecting others—in many investigations.

"It would also be an index-number on which, if people were so inclined, they could make contracts in a way analogous to the contracts for the commutation of tithe; in which the tithe is made to vary according to the prices of corn. To make the index-number useful for this purpose an Act would have to be passed prescribing the way in which the prices are to be obtained and published, and defining and giving a form for the contracts which might be made for payments, to vary according to the variation in the aggregate index-number. This would be a practical Tabular Standard such as Joseph Lowe, Jevons, and lately Professor Marshall, have suggested.

"All such index-numbers are liable to the observation that innumerable articles are, and must be, in the nature of things, wholly excluded. The variety of small articles is almost infinite. The assumption may also be made, I think, that on balance the permanent tendency is for such articles on the average, through the progress of invention, to increase in aggregate importance in proportion to the other articles which can be got into an index-number and, at the same time, individually to fall relatively in price. In investigations general facts of this kind would, of course, have to be borne in mind as qualifying deductions based upon the precise figures which the index-numbers may give. People making contracts based on index-numbers would also require to study what the effect would be likely to be on the result they wish to arrive at."—*End of Extract.*]

ANALYSIS OF CONTENTS.

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The usefulness of our result will be enhanced by an estimate of its accuracy. It would be desirable, if possible, to ascertain a numerical limit which the *error* [1] incurred by our calculation cannot possibly, or at least with any reasonable probability, exceed. But it is doubtful whether such a limit admits of being fixed with precision. The erroneousness of the conclusion could only be ascertained by inference from the inaccuracy of the premises. But it is difficult to appreciate with mathematical precision the error to which our data are liable. We may, however, argue that, if the erroneousness of the premises is approximately of a certain amount, if the error of the data is of a certain *order*, then the error of the conclusion will be of a certain other order.

[1] The use of the term "error" to denote a deviation from an unknown ideal is somewhat infelicitous. But the advantage which the term has in being familiar to the student of Probabilities may, it is hoped, preponderate over the disadvantage that it suggests to the general reader a more gross, blameworthy, and avoidable mistake than is contemplated here.

The subject of our investigation being thus defined, we may show that the erroneousness of the result is *less* than that of the data. There are two lines of proof converging to the truth of this theory. *First*, we may reason *a priori* by the Calculus of Probabilities that the index-number is subject to a smaller percentage of error than the weights and relative prices (given or referred to in columns 5 and 6 of the table). *Secondly*—this deduction may be verified by actual trial. We may assign a certain set of weights and relative prices as correct, and construct several sets of variants diverging from the "correct" figures in haphazard fashion. Then, operating with each set of variant data, we may calculate several variant index-numbers. These, it will be found, diverge less—that is by a smaller percentage—from the correct index-number than any set of variant data from the corresponding correct datum.

The second part of the evidence cannot be fully appreciated without the prior reasoning. By itself it conveys only a moiety of the truth. Those who are content with that fraction of knowledge are advised to skip the reasoning of the immediately following paragraphs, and to pass on to the more easily read lessons of experience (at p. 312 below).

The index-number which is the result of our calculations is subject to a less error than the data which enter into it, for two reasons. *First:* The numerator and denominator of the fraction which constitutes the index-number form each an aggregate of elements or parts, whereof each element is subject to a presumably independent error. Now, by a well-known principle of the Calculus of Probabilities, the percentage error of such an aggregate is less than the percentage error incident to each element (or at least to an element of average erroneousness). This principle applies to the errors both of the *weights* and the *observations* (relative prices).* The next consideration applies only to the former class of data. An error in any *weight* affects both the numerator and denominator in the same direction, whether of excess or defect, and thus is to a certain extent self-corrected.

This reasoning may be exhibited more fully by the aid of symbols. Let us put the series p_1, p_2, etc. . . . p_n for the *real* relative prices. These relative prices may be conceived as percentages obtained after the manner of Mr. Palgrave (see Table 26 of Memorandum in Appendix to "Third Report of the Commission on Depression of Trade") by multiplying the ratio

* The term "price-variation" was employed for what is now substituted "relative price."

$\frac{\text{New price}}{\text{Old price}}$ by 100. Let us denote the *apparent* relative prices, the erroneous observations, as $p_1(1 + e_1)$, $p_2(1 + e_2) \ldots p_n(1 + e_n)$, where e_1, $e_2 \ldots e_n$ are each positive or negative errors, usually proper fractions. Similarly let w_1, w_2, etc., be the *real* weights; and $w_1(1 + \epsilon_1)$, $w_2(1 + \epsilon_2)$ + etc., be the apparent, or erroneous, weights.

The index-number obtained from such data is

$$\frac{w_1(1 + \epsilon_1) \times p_1(1 + e_1) + w_2(1 + \epsilon_2) \times p_2(1 + e_2) + \text{etc.}}{w_1(1 + \epsilon_1) + w_2(1 + \epsilon_2) + \text{etc.}}.$$

Alike in the numerator and denominator of this expression we may segregate the *correct* and the *erroneous* portion; and reason by the first of the principles above mentioned that the incorrect portion is of a smaller order than the sum of the correct terms (the number of observations being sufficiently great). Accordingly it will be allowable to expand by Taylor's Theorem and neglect higher terms. We shall thus obtain a simple expression for the error of the resultant index-number in terms of the errors to which each class of the data is liable.

This investigation may be broken up into three steps: we may consider successively three cases in an order of increasing complexity. First (1) we shall suppose that the weights only are liable to error. Then (2) we shall introduce the circumstance that the observations, the relative prices, are themselves incorrect. Lastly (3) we shall take account of the fact that certain categories of articles may be altogether unrepresented.

(1) Under the first head we shall first consider the simple case when the weights are really equal, though apparently somewhat unequal. In this preliminary case the symbolic expression above written becomes simplified by the disappearance both of the e's and the w's. Expanding and segregating the heterogeneous elements in the manner indicated, we may write our result thus :—

$$\frac{p_1 + p_2 + \text{etc.}}{n}\left\{1 + \frac{p_1\epsilon_1 + p_2\epsilon_2 + \text{etc.}}{p_1 + p_2 + \text{etc.}} - \frac{\epsilon_1 + \epsilon_2 + \text{etc.}}{n}\right\},$$

where the term outside the brackets is the *correct* index-number, and the difference of the second and third terms within the brackets is the error of the index-number: the relative error, as it may be called, or (if multiplied by 100) the percentage error, in symbols $\frac{\Delta \mathrm{I}}{\mathrm{I}}$, if I is the correct index-number. The result obtained may be written

$$\frac{\Delta \mathrm{I}}{\mathrm{I}} = \frac{\mathrm{S}p}{n}\left\{1 + \epsilon_1\left(\frac{p_1}{\mathrm{S}p} - \frac{1}{n}\right) + \epsilon_2\left(\frac{p_2}{\mathrm{S}p} - \frac{1}{n}\right) + \text{etc.}\right\}.$$

In this expression call the factors of ϵ_1, ϵ_2, etc., respectively $\frac{1}{n}E_1$, $\frac{1}{n}E_2$, etc. Then $\frac{\Delta I}{I}$, the error whose magnitude we have to estimate, is $\frac{1}{n}(E_1\epsilon_1 + E_2\epsilon_2 + \text{etc.})$. To determine the probable and improbable limits of this quantity we require to know the magnitude, or at least the average extent, both of the E's and the ϵ's. The former datum depends upon the dispersion of the observations (the comparative prices) about their mean. For any E, *e.g.*—

$$E_r = n\left(\frac{p_r}{Sp} - \frac{1}{n}\right) = n\frac{\left(p_r - \frac{Sp}{n}\right)}{Sp} = \frac{\left(p_r - \frac{Sp}{n}\right)}{\frac{Sp}{n}}$$

= the deviation or *error* incurred by the individual relative price as compared with the average of a whole set; *relative* to (divided by) the average. Such a deviation might be symbolised as $\frac{\Delta p}{p}$, if we put p for the average relative price.

We may now proceed in two ways: (α) we may either suppose the deviations E_1, E_2, ascertained for the particular year or epoch to which the calculation in hand may refer; (β) or we may seek a measure for general use, and available without the trouble of examining the dispersion of the relative prices for a particular year. In either case we are to regard the ϵ's as errors grouped in random fashion about a mean, which is zero. The coefficient which measures the dispersion of these errors, the *modulus* for the ϵ-fluctuation, must be supposed knowable. Call it κ.

(α) On the former understanding, we are to regard E_1, E_2, etc., as known factors. Accordingly by a well-known theorem the *modulus*, which measures the extent of the error

$$\frac{1}{n}(E_1\epsilon_1 + E_2\epsilon_2 + \text{etc.}) = \frac{1}{n}\sqrt{E_1^2 + E_2^2 + \text{etc.}} \times \kappa,$$

$$= \frac{1}{\sqrt{n}}\sqrt{\frac{E_1^2 + E_2^2 + \text{etc.}}{n}} \times \kappa,$$

(β) Otherwise we are to regard E_1, E_2, as samples, so to speak, taken from an indefinite number—a complete series (in Dr. Venn's phrase) of E's. We must suppose the coefficient of fluctuation, or modulus, for this series to be given by prior experience. Let it be C. Then we may put as the most probable value for the measure or *modulus* of $\frac{\Delta I}{I}$, the error under consideration,

$$\frac{1}{\sqrt{n}} \times \frac{C}{\sqrt{2}} \times \kappa.$$

But this *most probable* measure may conceivably not be the *best* measure.* We must take into account that the real measure *may* be larger, and accordingly that, by adopting the measure described as "most probable," we may be underrating the probability of each extent of deviation (from zero) to which the quantity $\frac{1}{n}[E_1\epsilon_1 + E_2\epsilon_2, \text{etc.}]$ is liable. However, the error thus introduced is only of the order $\frac{1}{\sqrt{n}}$, that is, the $\frac{1}{\sqrt{n}}$th part of the magnitude to be evaluated. Now that degree of error has been already incurred by the neglect of the higher terms in the expansion of $\frac{\Delta I}{I}$. Accordingly it would be nugatory to apply correctives to the error now under consideration.

We have now to introduce the circumstance that the weights, both real and apparent, differ from unity. It may be shown that in the new expression for $\frac{\Delta I}{I}$ the coefficient of any weight-error ϵ_r is $\frac{w_r p_r}{Swp} - \frac{w_r}{Sw}$; which may be put in the form $\frac{w_r}{Sw}E'_r$, where E'_r is now the proportional deviation of p_r from the *weighted* mean of the p's; viz. $\frac{Swp}{Sw}$. Accordingly the modulus of $\frac{\Delta I}{I}$ becomes

$$\frac{\sqrt{w_1^2E'^2 + w_2^2E'^2 + \text{etc.}}}{Sw}\kappa.$$

In evaluating the coefficient of κ there are, as before, two courses. Either (α) we operate upon the known values of E'_1, E'_2, etc., for the particular year or epoch with which we are concerned. Or (β) we may make a general estimate based upon several years' experience, and roughly applicable to the unexamined data of any year.

(α) In the former case there is nothing more to be said, except that it will be legitimate in the evaluation of the modulus to put for w_1, w_2, etc., their *apparent* values; which may be written $w_1 + \Delta w_1$, $w_2 + \Delta w_2$, etc. For the error thus introduced into the modulus is of a negligible order.

(β) The general expression in terms of the E-fluctuation is found by considering that the most probable value of the quantity under the radical sign in the last written expression is $\sqrt{(w_1^2 + w_2^2 + \text{etc.})}\frac{C^2}{2}$, where $\frac{C^2}{2}$ is the mean square of error

* Or is it sufficient to say that, C and κ being uncorrelated, the expectation of their product = the product of their expectations? (cp. *Ency. Brit.*, Art. "Probability," § 15).

measured, not, as before, from the simple (arithmetical) mean (of many batches of p's), but from the weighted mean $\frac{Swp}{Sw}$; a difference which may be shown as follows to be of an order which may for our purpose be neglected.

The deviation of any p from the Weighted Mean—the relative or proportionate deviation—E'

$$= \frac{\frac{Spw}{sw} - p_r}{\frac{Spw}{sw}}.$$

This ratio may be thus expressed in terms of E_r, the deviation of p_r from the *Simple Arithmetic* Mean. Put v for the difference between the weighted and simple means. Then we have

$$E'_r = \frac{\left(\frac{Sp}{n} - p_r\right) - v}{\frac{Sp}{n} - v} = \frac{E_r - \frac{v}{p}}{1 - \frac{v}{p}},$$

if we put p for the Arithmetic Mean of the p's.

Now $v = \frac{Sp}{n} - \frac{Spw}{Sw} = \frac{p_1 + p_2 + \text{etc.}}{n} - \frac{p_1 w_1 + p_2 w_2 + \text{etc.}}{w_1 + w_2 + \text{etc.}}$.

Substitute for p_r its value $p\,(1 + E_r)$ (where p is the Arithmetic Mean of the p's); and we have

$$v = p\left[\frac{E_1 + E_2 + \text{etc.}}{n} - \frac{w_1 E_1 + w_2 E_2 + \text{etc.}}{w_1 + w_2 + \text{etc.}}\right]$$

$$= \frac{1}{n} p E_1 \frac{\frac{Sw}{n} - w_1}{\frac{Sw}{n}} + \frac{1}{n} p E_2 \frac{\frac{Sw}{n} - w_2}{\frac{Sw}{n}} + \text{etc.}$$

Put for the relative deviation of any w from the Arithmetic Mean of all the w's (the coefficient of $\frac{1}{n} p E_r$ in the last written expression) η_r. Then we have

$$v = \frac{1}{n} p [E_1 \eta_1 + E_2 \eta_2 + \text{etc.}].$$

The expression in brackets hovers about the value zero according to a law of error whose modulus is $\frac{\sqrt{n} C \chi}{\sqrt{2}}$; where C, as before, is the modulus of the E's, and $\frac{\chi^2}{2}$ is the mean square of the η's. Hence $\frac{v}{p}$ is of an order $\sqrt{n}$ times smaller than $C\chi$.

Now from the equation connecting E′ and E it appears that the sum of squares $E'^2_1 + E'^2_2 +$ etc. which occurs in the complete expression for the modulus of $\frac{\Delta I}{I}$ may be written

$$\frac{SE_r^2 + n\left(\frac{v}{p}\right)^2}{\left(1 - \frac{v}{p}\right)^2};$$

whence, as $SE_r^2 = n\frac{C^2}{2}$, it appears that the influence of $\frac{v}{p}$ may be neglected, n being supposed large.

We may therefore write

$$\text{Modulus of } \frac{\Delta I}{I} = \frac{\sqrt{Sw_r^2}}{Sw} \frac{C}{\sqrt{2}}\kappa;$$

or, employing the notation which we had lately occasion to introduce :

$$\text{Modulus of } \frac{\Delta I}{I} = \frac{1}{\sqrt{n}} \times \sqrt{1 \times \frac{\chi^2}{2}} \times \frac{C}{\sqrt{2}} \times \kappa.$$

This formula may be employed to utilise present as well as past experience. If we treat $\frac{\chi^2}{2}$ and $\frac{C^2}{2}$ as respectively the mean square of deviation obtained from the set of weights and price-returns entering into the index-number which we are computing, we shall thus have an approximate formula more convenient than the complete expression for the Modulus.

(2) We have now to introduce the circumstance that each p is liable to an error pe. Each element of error of the form $\Pi\epsilon_r$ is now aggravated by an element of the form $P e_r$. Accordingly the modulus of the total error will be $\sqrt{\Pi^2\kappa^2 + P^2c^2}$, where κ and c are the moduli for the independent partial errors of the weights and the prices respectively, Π is the coefficient of κ in the expression for the modulus of $\frac{\Delta I}{I}$ in case (1) and P^2 is equal to $\frac{Sw_r^2p_r^2}{(Swp)^2}$.

There may now be required, as before, a general formula applicable without any examination of the prices and weights on a particular occasion; or without other data than the coefficients expressing the dispersion of the prices and weights respectively. With this view, employing the notation already explained, and rejecting terms which may be shown to be of an inferior order, we may put for

$$\frac{Sw_r^2p_r^2}{(Swp)^2} \text{ the expression } \frac{1}{n}\left(1 + \frac{\chi^2}{2}\right)\left(1 + \frac{C^2}{2}\right).$$

Hence for the modulus of $\frac{\Delta I}{I}$ in the general case we have

$$\frac{1}{\sqrt{n}} \times \sqrt{1 + \frac{\chi^2}{2}} \times \sqrt{\frac{C^2}{2}\kappa^2 + \left(1 + \frac{C^2}{2}\right)c^2}.$$

(3) So far we have been estimating the errors due to the weights and prices of the articles which enter into our index-number not being accurate. We have now to take into account that not only are all those articles misrepresented, but also that certain other articles may be wholly unrepresented. For it is unlikely that all the classes of products which ought by rights to enter into an index-number can, even constructively, put in an appearance.

We have now to superinduce the error due to such omission upon the errors already estimated. To effect this we proceed in the same way as when compounding the errors proper to our first and second headings. That is, we shall separately evaluate for the third species of error its modulus squared, or *fluctuation*, as the present writer has proposed to term this important coefficient. Then we shall add the third fluctuation to the sum of the two preceding: that is, to the square of the formula given at the end of the second heading.

To find the fluctuation proper to the third heading, let us begin with the simple case in which the weights are all equal. As before, let Sp represent the sum of the observed (comparative) prices; let n be their number; and for $\frac{Sp}{n}$ put simple p. Let S$'p$ be the sum, and n' the number, of the *unobserved* prices. Then the error incurred by putting p for the Mean of all the prices, the relative error $\frac{\Delta I}{I}$, is

$$\left\{\frac{Sp + S'p}{n + n'} - \frac{Sp}{n}\right\} \div \frac{Sp + S'p}{n + n'}.$$

The most probable value of this expression is zero; while its *fluctuation* is found to be, in terms and by methods already explained,*

$$\frac{1}{n^2} \times \frac{2nn'}{(n + n')} \times C^2.$$

Now superadd the circumstance that the weights are various, dispersed about their mean according to the modulus χ; and connect the resulting expression with the square of the formula given at the end of heading (2).

The formula will require modification, if there is reason to

* That is treating the supposed complete set of observations obtained at one time as a *specimen* of a series obtained at other times (*cp.* above, p. 215). Otherwise we may regard Sp + Sp' as the "universe," of which Sp constitutes a *sample*

believe that the omitted articles have not the same average weight as those which are included; for instance, if, as is likely, the omissions are many in number, but inconsiderable in weight.

It will be noticed that in passing from (the dispersion of) the observed prices and weights to what has not been observed there is an inductive hazard greater than is involved by solutions of cases (1) and (2) in their more exact form, and while we suppose (as in the examples which will be adduced below) that the errors of weight and price emanate from regular and stable sources, so as to admit of safe prediction.

As in case (2), we may suppose the coefficients χ and C based either on prior experience or on the data appertaining to the particular calculation which is in hand.

It will be observed that these coefficients do not contribute equally to the resultant error represented by our formula. C, expressing the dispersion of the prices, is more efficacious than χ, appertaining to the weights. Similarly c, the measure of the error incident to the prices, affects the error of the index-number more than κ, the corresponding modulus of the weights.

It is proposed now to illustrate the formulæ which have been given by working a few examples. In these examples the statistical materials, the prices and weights, are taken out of Mr. Palgrave's Memorandum, from Tables 26 and 27 respectively. The conjectural arbitrary assumptions which will be made are that any price, and likewise any weight, is as likely as not to be out, in excess or defect, of the true figure by 10 per cent., but very unlikely to be out by 40 per cent., or, more exactly, that the apparent values fluctuate about the real one in conformity with a modulus which is 21 per cent.

Of the immense variety of cases which might be constructed by combining in different ways the attributes which define the preceding paragraphs, it will be sufficient here to discuss the most important case (2) of both weights and prices subject to error—divided into two species, according as (α) we utilise all the data special to the calculation in hand, or (β) content ourselves with the more summary estimate.

Let us apply these tests to Mr. Palgrave's computation of a weighted mean for the year 1885 (Memorandum in Appendix to *Third Report on the Depression of Trade*, 1886, C. 4797). First, according to method (α), the expression for the (proportionate) error due to a particular element of the index-number, the weight and price of a particular commodity, is

$$e_r \frac{w_r}{\mathrm{S}wp}\left[p_r - \frac{\mathrm{S}wp}{\mathrm{S}w}\right] + e_r \frac{w_r}{\mathrm{S}w} p_r.$$

Whence, as the Modulus of the error to which the computed index-number is liable, we have—putting p' for the *weighted mean* of the price-returns, and remembering that c and κ are the Moduli of the errors e and ϵ respectively—

$$\frac{1}{Swp}\sqrt{Sw_r^2(p'-p_r)^2\kappa^2 + Sw_r^2 p_r^2 c^2}.$$

The w's are given in Mr. Palgrave's column headed "Relative Importance" (Table 27, year 1885, p. 35). The p's are to be extracted from his Table 26. The weighted mean p' is, according to him, 76. And Swp is the sum of his column (for the year 1885), headed "Comparative," etc., *multiplied by* 100; that is 166,900. The rest of the expression above written is evaluated in the following table; of which the materials are taken from the sources named. The third column is formed by subtracting from each of the entries for 1885 in Mr. Palgrave's Table 26—*e.g.*, 38 the price of cotton (comparative with 1865–9)—the weighted mean 76. The last three columns in Mr. Palgrave's Table 26, relating to *Cotton Wool*, *Cotton Yarn*, and *Cotton Cloth*, are omitted, as they do not figure in his Table 27, and, it may be added, cannot be supposed *independent* of the price of cotton. The last column in our table is formed by squaring each entry in Mr. Palgrave's column headed "Comparative," etc. (Table 27, year 1885), and omitting the last digit :—

No. of Article.	Name of Article.	w.	w^2.	$(p_r - p')$.	$(p' - p_r)^2$.	$w^2(p' - p_r)^2$.	$w_r^2 p_r^2$.
			00's omitted			00,000's omitted	00,000's omitted
1	Cotton	263	691	−38	1,444	998	1,000
2	Silk	12	1	−23	529	0	4
3	Flax, etc.	49	24	−15	225	0	90
4	Wool	142	202	− 7	49	10	980
5	Meat	524	2,745	+26	676	1,855	28,622
6	Iron	150	225	+ 6	36	8	1,510
7	Copper	39	15	−27	729	11	53
8	Lead	13	2	−19	341	1	5
9	Tin	15	23	+ 2	4	0	14
10	Timber	164	269	+31	961	258	3,099
11	Tallow	28	8	+ 8	64	0	58
12	Leather	80	64	+34	1,156	73	774
13	Indigo	5	0	+35	1,225	0	4
14	Oils	49	24	− 7	49	1	116
15	Coffee	8	1	−14	196	0	3
16	Sugar	149	223	−23	576	128	624
17	Tea	71	50	− 7	49	2	240
18	Tobacco	29	8	+27	729	6	90
19	Wheat	410	1,681	−16	256	430	5,856
Sums		2,200	6,256			3,781	43,142

According to the hypotheses above made let us put c and κ each $= \cdot 21$. Then for the sought Modulus we have

$$\cdot 21 \frac{\sqrt{378,100,000 + 4,314,200,000}}{166,900} = \cdot 21 \times \cdot 41 \text{ (nearly)}.$$

Thus the error incident to each of the data has been reduced by more than a half in the result. It may be observed that the prices contribute much more largely than the weights to the total error. If we reduce the error incident to each price-return by a half, making its modulus ·1, instead of ·21, the total error of the result will be reduced by nearly a half—from modulus ·086 to modulus ·046. If we suppose the price-returns to be quite correct, then the error of the result due to the weights alone would be nearly half as small again, namely, of modulus ·025. This is agreeable to what was said above, that an error of the prices affecting only the numerator of the index-number is not, as in the case of the weights, compensated by an error affecting the denominator in the same degree.

Let us see now (β) how we should have fared if we had based our estimate on the grouping of the weights and prices in prior experience, such as is afforded by the table of prices cited from the *Economist*.

The dispersion of the price-returns, the coefficient C in the general formula, is thus to be found—in the case of the year 1884 for example. The arithmetic mean of the first nineteen entries in Table 26 for 1884 is 81 nearly. The "differences" and squares of differences are computed in the accompanying table. The

Name of Article.	Price.	Differences. −	Differences. +	Squares of Differences.
Coffee . . .	70	11		121
Sugar . . .	77	4		16
Tea . . .	81	0		—
Tobacco . .	90		9	81
Wheat . . .	73	8		64
Meat . . .	103		22	484
Cotton . . .	37	44		1,936
Silk . . .	66	15		225
Flax, etc. . .	59	22		484
Wool . . .	73	8		64
Indigo . . .	107		26	676
Oils . . .	81	0		—
Timber . . .	105		24	576
Tallow . . .	109		28	784
Leather . .	106		25	625
Copper . . .	70	11		121
Iron . . .	76	5		25
Lead . . .	61	20		400
Tin . . .	90		9	81
Sums . . .	...	148	143	6,763

mean square of difference, 353, divided by the square of the mean 6561 forms an approximate, a *prima facie* value for $\frac{C^2}{2}$, namely, ·04.

$$\text{Mean square of deviation} = \frac{6765}{19} = 353.$$

For the year 1880, taken similarly as a random specimen, the mean (of the nineteen prices) is found to be 93·5, and the mean square of differences 434. Accordingly the value for $\frac{C^2}{2}$ is ·05. Proceeding similarly for 1873, another year taken at random, we find for $\frac{C^2}{2}$ again ·05. As the mean of the three values we may put ·05.

To find the dispersion of the w's we proceed similarly. The arithmetical mean is for every year 2200 ÷ 19, or 116 nearly. The " differences " are to be formed by subtracting this figure from each of the entries in the column headed *Relative Importance*, in Mr. Palgrave's Table 27. The sum of the squares of the differences is to be divided by 19 for the absolute mean square of difference as it may be called. This result, divided by 116^2, gives the mean square of deviation relatively to the mean weight. The values thus extricated for the years 1873, 1880, and 1884 respectively are, in round numbers, 354,000, 351,000, 357,000: each divided by 255,664 ($= 19 \times 116^2$); whereof the mean value is 1·38.

Substituting in the general or summary formula given under head (2) for the modulus of $\frac{\Delta I}{I}$ the values for C^2 and χ^2 just ascertained, and for c and κ the assumed value ·21, we have

$$\frac{1}{\sqrt{19}} \times \sqrt{2{\cdot}38} \times \sqrt{{\cdot}05 \times {\cdot}044 + 1{\cdot}05 \times {\cdot}044} = \frac{1}{4{\cdot}36} \times 1{\cdot}54 \times {\cdot}22 \text{ (nearly)} = {\cdot}077;$$

whereas the answer found by the more exact method was ·086. This consilience seems greater than might have been expected, considering the small number of the elements entering into the computation, only nineteen; and the scantiness of the induction by which we determine the coefficients C and χ.

If we employ the summary formula as a short method of utilising the data special to the index-number of 1885, we shall find that $\frac{C^2}{2}$ as based upon the fluctuation of prices for this year is ·08; and $\frac{\chi^2}{2}$ the mean square of deviation for the w's is still 1·38. Hence, as the approximate expression for the modulus, we have

$$\frac{1}{4{\cdot}36} \times 1{\cdot}54 \times {\cdot}21\sqrt{1{\cdot}16} = {\cdot}08.$$

Thus we reach much the same result by the shorter as by the more tedious route.

We shall presently—in the portion of this paper addressed to the general reader—try an experiment calculated to verify our deductive reasoning—so far as a theorem in the Calculus of Probabilities can be verified by a single experiment. We shall affect each of the elements in Mr. Palgrave's index-number for 1885, each weight and price, with a figure taken at random from a series of figures hovering about unity in conformity with a modulus equal to ·21. Such a series the writer happens to have ready to hand: consisting of sums of twenty digits taken at random from mathematical tables, where the mean value is 90 and the absolute modulus 19. The relative modulus, therefore, the modulus for the series when we divide each aggregate by 90, is ·21. Accordingly it will be sufficient to multiply each weight both in the numerator and the denominator with one of the sums (of twenty digits) taken at random, and similarly affect each price entering into the numerator, while the denominator is multiplied by 90.

To resume now, in popular language, this somewhat technical inquiry. The subject under investigation is the error to which our computation of index-numbers is liable—the error relative to, or per cent. of, the true value which we seek. We want to know, for instance, whether it is as likely as not that our calculation exceeds (or falls short of) the correct result by 10 per cent. of that result; whether it is very improbable that the excess (or defect) should be as great as 25 per cent.

The error thus conceived is found to depend in a definite manner upon *six* distinct circumstances. The erroneousness of the result is greater, the greater the inaccuracy of the data: the weights and the (comparative) prices. The erroneousness of the result is also greater, the greater the inequality of the weights, and the greater the inequality of the price-returns. Lastly, the result is more accurate, the greater the number of the data and the smaller the number of omitted articles.

These circumstances are not all equally operative. Other things being the same, the inaccuracy of the price-returns affects the result more than inaccuracy of the weights; and the inequality of the price-returns more than the inequality of the weights.

The only proof of the theory which can be offered to the unmathematical reader is to verify it by actual trial. We may assume a certain set of data as perfectly correct: then affect each of them with an error such that the modified datum is, say, as likely as not to be in excess or defect by 10 per cent.; is very unlikely to be out by 30 or 40 per cent.; and cannot, humanly speaking, be out by more than 50 per cent. A simple method of affecting a given set of figures with an error of this degree is to multiply each of them with a figure formed by adding together twenty digits taken at random from mathematical tables or statistical returns; dividing each product by 90 (the mean about which aggregates of twenty random digits hover). The data thus artificially affected with error are now to be used in the computation of an index-number, an erroneous number, which is to be compared with the result assumed to be true as having been deduced from the unfalsified data. A great number of such trials having been made, it will appear that the erroneous index-numbers deviate from the true one with the frequency and to the extent predicted by theory.

A specimen of this verificatory process is given on p. 318. The data employed by Mr. Palgrave in his computation of an index-number for 1885 [1] are assumed to be correct; then each datum is displaced or falsified in the manner above described, and a new (erroneous) index-number is deduced.

In this table the first column contains the names of articles in the order adopted by Mr. Palgrave in his Table 27. The second column contains the "weights" assigned by him under the heading of "Relative Importance." The third column consists of multipliers formed by adding twenty digits at random, and thus calculated to deflect the weights from their respective true values to the extent of, say, 12 per cent. on an average. The fourth column gives the new system of weights thus affected with error. The fifth column contains (comparative) prices taken from Mr. Palgrave's Table 26 for the year 1885. The sixth column furnishes a new set of multipliers assigned by chance. The seventh column gives the prices affected by error, and multiplied by 90 (the average value of the chance-multipliers). The eighth column gives the product of the erroneous weights and the erroneous prices (×90). The sum of this last column, 1,413,470,000, divided by ninety times the sum of the erroneous

[1] See p. 318.

weights, which sum is 172,486, gives the erroneous index-number 81; whereas the true index-number, on the assumption here made that Mr. Palgrave's data are absolutely correct, is, as computed by him, 76.[1]

Thus the falsified result is too great by $\frac{5}{76}$, or about 6 or 7 per cent. That is a result quite consonant with the theory which assigns such a *measure* of the error to be expected [2] that the result is as likely as not to be out by 4 per cent., and that the odds are only five to one against the error being so large as 8 or 9 per cent.

Articles.	Real weights.	Sums of twenty digits.	Apparent weights.	Real prices.	Sums of twenty digits.	Apparent prices × 90.	Apparent weights × apparent prices, 0,000's omitted.
Cotton . . .	263	81	18903	38	82	3316	6268
Silk . . .	12	69	828	53	99	5247	434
Flax, etc. . .	49	97	4753	61	97	5917	2812
Wool . . .	142	80	11360	69	81	5589	6349
Meat . . .	524	68	35632	102	74	7548	26913
Iron . . .	150	81	12150	82	88	7216	8767
Copper . . .	39	87	3393	59	87	5133	1739
Lead . . .	13	66	858	57	85	4845	415
Tin . . .	15	85	1275	78	104	8112	1034
Timber . . .	164	71	11644	107	110	11770	13705
Tallow . . .	28	87	2436	84	94	7896	1924
Leather, etc. . .	80	110	8800	110	84	9240	8131
Indigo . . .	5	74	370	111	110	12210	4518
Oils . . .	49	89	4361	69	69	4761	20805
Coffee . . .	8	85	680	62	62	3844	2613
Sugar . . .	149	80	11920	53	109	5777	6886
Tea . . .	71	96	6816	69	79	5451	3438
Tobacco . . .	29	93	2697	103	89	9167	2478
Wheat . . .	410	81	33210	60	111	6660	22118
Sums . . .	—	—	172086	1427	1714	129699	141347
	—	—	—	75·1	—	75·7	81

It would have been nothing miraculous if the result had been out by *sixteen* per cent.; nothing more extraordinary than, for instance, the fortuitous sequence which may be observed in our third column of *eight* random aggregates falling below the average about which they should oscillate, namely, 90.[3]

The same table furnishes another verification, if, making

[1] *Third Report on Depression of Trade*, Appendix B. Memorandum by R. I. Palgrave. Tables 26 and 27.

[2] Taking 8·5 as the *Modulus* of the resultant error. See above, p. 315.

[3] The probability of an error exceeding 1·9 times its modulus is ·0072. The probability of the sequence referred to is $\cdot 0078 \left(= \frac{1}{2^7}\right)$.

abstraction of Mr. Palgrave's weights, we assume the index-number calculated on the principle of the economist to be correct, and regard the figures in our sixth column as erroneous weights (the true weights being all equal). Upon this understanding we have the true result, the Simple Arithmetic Mean of the comparative prices, 75·1; whereas the erroneously Weighted Mean is 75·7, that is, it is in excess by about ·8 per cent. Now the measure of error here predicted by theory[1] is such that an error of ·7 per cent. is as likely as not to occur. The occurrence of ·8 per cent. is therefore eminently consonant with the theory.[2]

It might be desirable to apply this sort of test on a large scale to the computation recommended by the Committee, and to prove by specific experience the conclusions which are deducible from the Theory of Probabilities concerning the accuracy of any index-number.

These conclusions cannot be stated in their most exact form until the price-returns, as well as the weights which enter into the computation to be tested, are assigned. But even at the present stage of our procedure, and without reference to the price-returns of a particular year, we may approximately estimate the accuracy of index-numbers of the kind proposed by the Committee. For the purpose of a rough estimate it is enough to know the weights (which are assigned in the Second Report of the Committee) and to utilise past experience concerning the course of prices in this country. A certain datum,[3] which had better be determined precisely from the price-returns from the particular year to which the index-number relates, may be approximately obtained by induction from the experience of past years.

Eliciting the required datum from the prices recorded by the *Economist*,[4] we may provisionally assert the following propositions concerning the accuracy of index-numbers such as the Committee

[1] By case (1) above, p. 306, the modulus is $\frac{1}{\sqrt{n}} \times \sqrt{\frac{\overline{SE_r^2}}{n}} \times \kappa$. Here n is 19; $\frac{SE_r^2}{n}$ is found to be ·08, and κ is ·21. Whence the modulus is about ·014, or 1·5 per cent.

[2] Perhaps it may be asked here whether the example given is suited to exemplify our estimate of the *third* species of error (see above, p. 306): that due to the total omission of certain articles. The answer is that this estimate, involving a larger element of induction, does not profess to be so amenable to verification as those which are derived from known and steady "sources of error," like our aggregates of digits. Moreover, such verification as the theory admits would require a larger number of items than the table in the text contains.

[3] The coefficient C defined above, p. 308.

[4] As given in Mr. Palgrave's Table 26 (see above, p. 313).

has proposed. These, it will be recollected, involve twenty-seven English price-returns and twenty-seven assigned weights.[1]

(1) In such an index-number, if the weights alone are supposed subject to error, then the average error of the result, its erroneousness as one may say, is *twenty* times less than the error to which each weight is liable.

(2) If the price-returns alone are liable to error, the erroneousness of the result is about *four and a half* times less than that of each datum.

(3) In the general case, when both prices and weights are liable to error, then, if that error be the same for both species of data, the error of the result is still about four and a half times less than that same. If the error of the weights become twice as great as that which is incident to the prices, other things being the same, the error of the result is not materially increased. The error of the weights would need to be *five* times as great as that of the prices in order to increase the error of the result by 50 per cent. (making it only *three* times less than the error incident to the prices alone).

The practical conclusion from these propositions appears to be : Take more care about the prices than the weights.

More detailed statements cannot be made without some assumption as to the degree of inaccuracy to which our data are liable, the extent to which our estimates of weights and prices deviate from the figures which would be assigned if our knowledge and theory were perfect. In entertaining any suppositions as to the extent of this discrepancy, it is proper to conceive that the larger deviations, the more extensive errors, are less frequent in the long run, or more improbable. Thus, if we suppose that a deviation of each datum, weight or price, to the extent of 10 per cent. is as likely as not, then it may be presumed that a deviation of 20 per cent. is not likely, of 30 per cent. very unlikely. Upon this hypothesis, according to the general formulæ above investigated, the error, or fortuitous deviation from the ideal, to which the Committee's index-number is liable is as likely as not to be as large as 2 or 3 per cent., but is unlikely to be 6 per cent., and very unlikely to be 10 per cent. Now let us entertain the more unfavourable and almost certainly extravagant hypothesis that each datum is as likely as not to be out by 25 per

[1] Namely, 5, 5, 5, 5; 10, $2\frac{1}{2}$, $7\frac{1}{2}$; $2\frac{1}{2}$, $2\frac{1}{2}$, 9, $2\frac{1}{2}$, 1, $2\frac{1}{2}$; $2\frac{1}{2}$, $2\frac{1}{2}$, $2\frac{1}{2}$, $2\frac{1}{2}$; 10, 5, $2\frac{1}{2}$, $2\frac{1}{2}$; 3, 1, 1, 3, 1, 1. Whence the value of $\frac{Sw_r^2}{(Sw)^2}$ (see above, p. 310) is found to be ·05.

cent., and may just possibly err to the extent of cent. per cent. (an error which, if possible *in excess*, is almost inconceivable *in defect*). Upon this hypothesis our index-number is as likely as not to be out 5 per cent. but is not likely to be out by 10, and very unlikely to be out by 15, per cent.

The presumption that our calculation is not likely to be far out is confirmed by comparing the results obtainable by our method with those obtained by other operators upon different principles. If the compared figures differ little from each other it is presumable that they differ little from the true, the ideally best, figure: that which would be obtained if the data were perfect.

The index-numbers which challenge comparison with those proposed by the Committee may be arranged under four categories, namely:

I. Those which are formed by taking the *Simple Arithmetical Mean* of the given relative prices; the principle of the *Economist's* index-number, or rather what would be the principle of that operation if the prices operated on had not been selected with some reference to the quantity of the corresponding commodities.

II. What may be called the *Weighted Arithmetical Mean*, each relative price being affected with a factor proportioned to the quantity of the corresponding commodity, the principle adopted by the Committee.

III. The *Geometric Mean*, as employed by Jevons.

IV. The *Median*, proposed by the present writer as appropriate to certain purposes.[1] It is (in its simplest variety) formed by arranging the given price-variation (*e.g.*, 98, 80, 88, 87, 85) in the order of magnitude (*e.g.*, 80, 85, 87, 88, 98) and taking as the Mean the *middle* figure (in the above example the *third* figure, *i.e.*, 87).

Under each of these headings it is desirable to supplement actual verification with *a priori* reasoning based on the principles laid down in the earlier part of the Memorandum.

We may begin with the case (A) in which the comparative prices are supposed the same for the compared index-numbers. Later on (B) we shall take examples in which both the comparative prices and the mode of combining them are different.

A.

I. Let us take the prices which are to hand for 21 (out of the 27) items of our index-number in Mr. Sauerbeck's well-known

[1] See Sect. IX. of the first Memorandum; above, p 247 *et seq.*

paper on the prices of commodities.[1] Let us form the Simple Arithmetic Mean of these prices for the year 1885, and compare it with the Mean obtained by applying our system of weights to the same prices. The operation is exhibited in the annexed table,

	1885.			1873.		
1	2	3	4	5	6	7
Articles common to Sauerbeck and the Committee.	Comparative Prices for 1885 given by Sauerbeck.	Weights assigned by the Committee.	Product of columns 2 and 3.	Comparative Prices for 1873 given by Sauerbeck.	Weights assigned by the Committee.	Product of columns 6 and 7.
Wheat . . .	60	5	300	108	5	540
Barley . . .	77	5	385	104	5	520
Oats . . .	79	5	395	98	5	490
Potatoes and rice .	67	5	335	116	5	580
Meat . . .	88	10	880	109	10	1090
Butter . . .	89	7½	668	98	7½	735
Sugar . . .	59	2½	147·5	101	2½	252·5
Tea . . .	64	2½	160	102	2½	255·5
Cotton . . .	62	2½	155	100	2½	250
Wool . . .	73	2½	182·5	118	2½	345
Silk . . .	55	2½	137·5	95	2½	237·5
Leather . .	94	2½	235·5	117	2½	292·5
Coal . . .	72	10	720	145	10	1450
Iron . . .	60	5	300	170	5	850
Copper . .	57	2½	142·5	112	2½	280
Lead . . .	57	2½	142·5	117	2½	292·5
Timber . . .	81	3	243	111	3	333
Petroleum . .	55	1	55	122	1	122
Indigo . . .	72	1	72	92	1	92
Flax . . .	73	3	219	97	3	291
Palm oil . .	77	1	77	97	1	97
Sums . .	1471	81·5	5952	2329	81·5	9395·5
Means . .	70		70·6	110·4		115

the latter columns of which present a similar comparison for the year 1873. The two results may thus be summed up :

	1885.	1873.
Simple Arithmetic Mean	70	110·5
The Committee's Weighted Arithmetic Mean .	70·6	115

[1] *Journal of the Statistical Society*, 1886.

The relation between these results is predictable by, and consilient with, the conclusions of *a priori* reasoning. Accordingly the inference that the deviation between the two computations is not likely to exceed a small percentage may safely be extended to adjacent cases.

It follows, from the principles laid down in the earlier part of this Memorandum, that the discrepancy to be expected between the two results depends on three circumstances: the number of items, the inequality of the relative prices, and the inequality of the weights. The measure or *modulus* of the discrepancy is, in our notation,

$$\frac{1}{\sqrt{2n}} \times C \times \chi,$$

where n is 21; C is presumed (by a sufficient, but certainly not very copious, induction) to be from ·2 to ·3; and χ is found to be about ·9.[1]

It follows that of the observed discrepancies, ·6 and 5, one is, *a priori*, more likely than not to occur, and the other not unlikely. A rapidly increasing improbability attaches to the higher degrees of divergence.

Of course it must be understood that this theorem in Probabilities, this statement of what will occur in the long run, is based upon the supposition that the weights are distributed impartially among the comparative prices. But if throughout the whole run the largest weight is attached to the largest, or smallest, observation, then the fortuitous character of the phenomenon is impaired. In fact the "long run" of which the theory may be expected to

[1] See above, p. 313, where the present writer records the *Mean Square of Deviation* for the comparative prices of nineteen different articles (given by the *Economist*) in different years. The *Mean Square of Deviation* for the figures given by Mr. Sauerbeck seems to be much the same. Again, the writer has, with much the same result, ascertained (by the Galton-Quetelet method) the quartiles for a few groups of English prices, like those given by Jevons. For example, in the case of the thirty-nine figures of the prices for prime articles in 1860–62 comparative with 1845–50 (*Currency and Finance*, pp. 51, 52) the quartile (half the interval between the tenth and the thirtieth) proves to be 11, corresponding to a modulus of about 22 per cent. If, however, we take in all the 118 articles given on the same page the quartile is 17. The groups of thirty-nine on Jevons' page 44, so far as they have been examined, give much the same result as the thirty-nine on pages 51, 52. Jevons himself gives $2\frac{1}{2}$ as the "probable error" incident to the *Mean* of thirty-six relative prices (*Currency and Finance*, p. 157)—corresponding to a probable error of 15, a modulus of 30 for the *individual* price-return. Doubtless the dispersion may be expected to be greater the more distant the base. If precision could be expected, it would be proper to express the coefficient as a percentage of the mean relative prices at each date rather than of the initial price or basis [as Bowley has done in the important Memorandum mentioned above, p. 198].

be true is a series of heterogeneous index-numbers not of consecutive years. Some imperfection of the sort noticed is observable in the case of Mr. Palgrave's Weighted Mean compared with the corresponding Simple Arithmetic Mean. The enormous weights attached to the continually low-priced *Cotton* and the continually high-priced *Meat* seem to affect the Weighted Mean abnormally. To effect the comparison, we must not take the averages given in Mr. Palgrave's Table 26, but those which are obtained by omitting from that table the three items *Cotton Wool, Cotton Yarn,* and *Cotton Cloth,* which do not occur in the compared Table 27. The annexed comparison does not present the appearance of pure chance. The discrepancies are rather *less* in magnitude than the theory requires. For the modulus, as deduced from Mr. Palgrave's system of weights, proves to be about 8·5 per cent. of the Mean 80 or 90 :[1] that is about 7, corresponding to a probable error of about 3·5. The set of differences above registered seems to range a little within the limits so defined.

	1870	1871	1872	1873	1874	1875	1876	1877	1878	1879	1880	1881	1882	1883	1884	1885
Mr. Palgrave's Weighted Mean for 19 articles	90	93	100	104	108	97	99	100	95	82	89	93	87	88	80	76
The Simple Arithmetic Mean for the same articles	94	95	102	107·5	107	92	99	101	93	82	93·5	86	89	85·5	81	75
Excess of Arithmetic over Weighted Mean	+4	+2	+2	+3·5	−1	−5	0	+1	−2	0	+4·5	−7	+2	−2·5	+1	−1

The reason is, doubtless, that the impartial sprinkling of the prices among the weights, presupposed by theory, is not fulfilled in fact. Had it happened that throughout the whole run all the largest weights had been attached to the articles whose prices were continually low, e.g., *cotton,* and (for the last few years at least) *silk* and *flax,* then the discrepancies (between the weighted and simple mean) would have been rather larger than theory predicts. Thus, for the year 1885 I make *silk* exchange weights with *meat,* and thus bring down the index-number to 64; a discrepancy from the Arithmetic Mean which, if continued—as it probably would be—from year to year, would be a little too great. Similarly, when *wheat* exchanges weight with *leather,* and *cotton* with *indigo,* the index-number works out to 92—a discrepancy of two moduli, which is much too large for a continuance.

This sort of abnormality is less likely to occur in the case of our scheme, where none of the weights are so large as some of Mr. Palgrave's. Still, before pressing the theory, it is proper to examine whether the larger weights—in our case those of *meat,*

[1] See end of last note.

fish, and *coal*—are, from year to year, coupled with extreme relative prices.[1]

Whenever law of this sort is discernible the doctrine of Chances hides its inferior light, which is serviceable only in the night of total ignorance. The pure theory of Probabilities must be taken *cum grano* when we are treating concrete problems. The relation between the mathematical reasoning and the numerical facts is very much the same as that which holds between the abstract theory of Economics and the actual industrial world—a varying and undefinable degree of consilience, exaggerated by pedants, ignored by the vulgar, and used by the wise.

1	2	3	4	5
Relative prices for 1886.	Weights assigned by Sauerbeck.	Product of columns 1 and 2.	Relative prices for 1873.	Product of columns 2 and 4.
60	11	660	108	1188
77	5·5	423	104	572
79	6	474	98	588
67	6	402	116	696
88	15·5	1364	109	1689·5
89	3	267	98	294
59	5·5	325	101	555·5
64	2	128	102	204
62	10	620	100	1000
73	7·5	537·5	118	885
55	1	55	95	95
94	8	752	117	936
72	13	936	145	1885
60	5	300	170	850
57	1	57	112	112
57	0·5	28·5	117	58·5
81	2	162	111	222
55	0·5	27·5	122	61
72	0·5	36	92	49
73	1	73	97	97
77	0·5	15	97	19
Sum . .	105	) 7642·5	—	12056·5
Mean . .	—	73	—	115

[1] The effect of large weights combined with high prices is strikingly shown in an index-number (attributed to Dr. Paasche) which is published in Conrad's *Jahrbücher*, Vol. XXIII. p. 171. There are twenty-two items, among which *Rye* obtains about thirty per cent. of the total weight, and the Cereals generally (between whose prices there is a certain solidarity) about seventy per cent. It is no wonder that in the year 1868, when the price of the Cereals was exceptionally high, the Weighted Mean should be 118, while the Simple Arithmetic Mean of the twenty-two comparative prices is only 104.

II. Next let us compare our result with that obtained by using some other system of weights, *e.g.*, Mr. Sauerbeck's. In the table on page 325, column 1 is the same as column 2 of the table on p. 322, containing Mr. Sauerbeck's prices. Column 2 gives Mr. Sauerbeck's weights (for 1885) reduced to percentages of the total weight assigned by him to the twenty-one articles which are common to him and the Committee. For example, 61 is the weight actually assigned by him to wheat. This, multiplied by 100, and divided by 559, the sum of all the weights assigned by him to the twenty-one articles, gives 11 (nearly).

The comparison between the two systems is presented in the accompanying summary. The slightness of the difference between the compared results might have been predicted by theory, and may be predicted safely of adjacent cases.

	1885.	1873.
The Committee's System of Weights . . .	70·6	115
Mr. Sauerbeck's System of Weights . . .	73	115

III. We come next to the index-number of Jevons: the Geometric Mean of the relative prices appertaining to a number of groups. The definition of these groups is not wholly irrespective of their importance to the consumer and producer. There is evinced more or less concern that each article of equal importance should "count for one" in the composition of the index-number. But Jevons does not affect precision of weight. *Pepper*, for instance, forms one of the constituent thirty-nine articles.[1]

The analogue of this operation for our materials appears to be the Simple Geometric Mean of the relative prices for each of the articles specified in our scheme; except, indeed, those to which a very small weight, namely 1, has been assigned. Accordingly *Petroleum, Indigo, Palm Oil,* and *Caoutchouc* may, with propriety, be lumped into one group, for which the mean relative price is to be ascertained geometrically. For the sake of comparison with Mr. Sauerbeck's result Caoutchouc (not recorded by him) may be omitted from this little group. The Mean of the group so constituted is to be placed along with the relative prices

[1] In the "Serious Fall," republished in *Currency and Finance*, p. 44. In the "Variation of Prices" (*ibid.*, p. 142) Jevons seems to have employed the practice of weighting rather more extensively. He says, "Several qualities of one commodity have been joined and averaged before being thrown as one unit into larger groups"—in the case of certain articles which are not very clearly indicated. For the period after 1844 the [unweighted] "average prices, as calculated from the price-lists of the *Economist* . . . were mostly used."

for the remaining eighteen articles common to us and Mr. Sauerbeck, and the Geometric Mean of all the nineteen is to be taken. It proves to be 69 presenting the comparison herewith exhibited.[1]

The Committee's Weighted Mean of 21 articles	70·6
The slightly adjusted Geometrical Mean of the same . . .	69

The slightness of this divergence is conformable to theory. For it has been shown that the Weighted Mean (of twenty-one articles) is not likely to differ very much from the Simple Arithmetic Mean of the same. And it may be shown that the Arithmetic Mean is not likely to differ very much from the Geometric when the number of price-observations is large, and if they are not very unequal. This proposition may be illustrated by the following figures, the first row of which is obtained by taking the Arithmetic Mean of the thirty-nine price-percentages given—by Jevons in his paper on a "Serious Fall," etc. (*Currency and Finance*, p. 44). The second row consists of the Geometric Means, as given by him at p. 46, for the same figures. The superior magnitude of the Arithmetic Mean will be noticed. This circumstance (which Jevons thought an advantage on the side of his procedure) could not be predicated of a *Weighted* Arithmetic Mean (such as our index-number), as compared with the Geometric :—

	1851.	1852.	1853.	1855.	1857.	1859.
Geometric Mean for 39 articles .	92·4	93·8	111·3	117·6	128·8	116
Arithmetic Mean for same . .	94·6	94·6	112·4	119	134	119

IV. We come now to the *Median*, which has been recommended by the present writer as the formula for the most objective sort of

Below 70.	Between 70 and 80.	Above 80.
. . . .	72 72 73 73	. . . .
Ten below 70	Median = 72	Seven above 80

[1] If we lump together *Barley* and *Oats* into one group, *Sugar* and *Tea* into another, and again *Copper* and *Lead*, the Geometric Mean of the sixteen returns thus presented is 70·2.

Mean between prices, not directed to any special purpose, such as the wants of the consumer or the difficulties of the producer, but more impersonal and absolute.

Of the twenty-one relative prices for 1885 given in the Table on p. 325 we have to take that which is the *eleventh* in the order of magnitude. To ascertain this we need not arrange *all* the figures in order. Having an inkling that the Mean is between 70 and 80, we shall find it sufficient to note the number of returns which lie outside those limits, and to write down in the order of magnitude only the returns which lie between 70 and 80. Thus, running our eye down the column of figures, we make a dot on the right for every return which is greater than 80, on the left for every one less than 70; and write down in the central compartment the figures which lie between 70 and 80 inclusive. Whence it appears that 72 is the figure eleventh in the order of magnitude: that is the Median.

1	2	3
Comparative Prices.	Precisions determined by mass.	Arbitrary precisions.
60	2	2
67	2	1
59	1·5	2
64	1·5	2
62	1·5	1
55	1·5	1
60	2	1
57	1·5	2
57	1·5	1
55	1	2
	16	**15**
72	3	2
72	1	1
73	1·5	1
73	1·5	1
77	1	1
77	2	2
79	2	1
	12	**9**
88	3	2
99	2·5	1
94	1·5	1
81	1·5	2
	8·5	**6**
	36·5	**30**

This is the Simple or Unweighted Median. There is a variety constituted by assigning special importance to those returns which we have reason to suppose are specially good representatives of the changes affecting the value of money. If, as in the writer's Memorandum often referred to,[1] we take mass * of commodity as the principle of ponderation, we shall have to proceed as follows with our twenty-one articles :—

As before, make three compartments for returns below 70, for those between 70 and 80, and for those above 80 respectively. Write down in the first and third compartments the returns in the order in which they occur (in any order); but in the central compartment in the order of magnitude.[2] In the second column of each compartment write the figures representing the relative *precision* assigned to each return. If these estimates of precision are based upon the quantities of commodity, it is recommended that they should be equal to, or rather less than, the square roots of the proportionate masses. Accordingly 2 has been put for the square root of 5, 1·5 for the square root of $2\frac{1}{2}$, and so on. Add together the sums of all the second columns. Thus, $16 + 12 + 8{\cdot}5 = 36{\cdot}5$. Find the central figure of the total second column : that is the figure which as nearly as may be has 18·25 for the sum of figures above it and below it. This figure proves to be the 3 at the top of the second compartment opposite 72. Then 72 is the required Mean.

In the third column another system of precisions has been tried to illustrate the effect of treating some relative prices as more typical of the change in the value of money than others. Tossing up a coin, the writer has stuck down (corresponding to each figure in the first column) 2 if heads turned up, 1 if tails. The sum of these arbitrary coefficients of precision is 30, and accordingly the adjusted Median is the point intermediate between 72 and the return next below in the order of magnitude, which proves to be 67. The adjusted Median is, therefore, 69·5.

By operating similarly on the price-returns for 1873 (given above) it is found that the Simple Median is 108, the Median adjusted by taking account of quantities still 108.

The deviation between the Median and the Simple (or other) Arithmetic Mean cannot, so far as the writer knows, be formulated

[1] Section IX. of the first Memorandum.

* "Commodity" may be understood as equivalent to "utility," when the term mass of commodity is put for volume of value representing quantity of satisfaction.

[2] It will probably be convenient to write these returns first in the order of their occurrence, and then rearrange them.

exactly. It diminishes with the number of observations, being of the order $\frac{1}{\sqrt{n}}$. A superior limit is given by a small multiple, say twice, $\sqrt{1 + \frac{1}{2}\pi}$ of Modulus of the observation; in our case of say ·1, or 10 per cent.[1] This limit is probably very superior, as the following trials, in addition to those given above, suggest :—

	1861.	1862.	1863.	1866.	1867.	1869.
Arithmetic Mean for 39 articles .	94·6	94·6	112·4	119	134	119
Median for the same . .	92	92	108	111	127	116·5
Geometric Mean for the same .	92·4	93·8	111·3	117·6	128·8	116

The thirty-nine figures are those above referred to, given by Jevons at p. 44 of his *Currency and Finance*. The Geometric Means have been cited again here in order to bring out the interest-

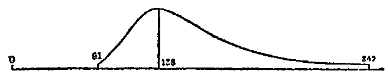

Fig. 1.

ing fact that the Median seems to keep closer to the Geometric than the Arithmetic. This property (which it would be desirable to verify more fully) is agreeable to the theory, first advanced by the present writer so far as he is aware, that prices are apt to group themselves in an unsymmetrical fashion after the pattern of the curve in Figure 1, whose ordinates indicate the frequency of each relative price. In the year 1857, for instance, the smallest figure was 91, the largest 247; while the Geometric, Median, and Arithmetic Means were respectively 129, 127, and 134. There is some reason to believe that the Geometric and Median—especially the latter—are more apt to be coincident with the point at which the greatest number of returns cluster, the greatest ordinate of the curve.

If then we take as our quæsitum *that figure* which *would be presented by the greatest number of relative prices* in the complete series of returns for all articles great and small, then, regarding our twenty-one, or it may be forty-five, articles as *specimens* of this series, we shall best operate on them by taking their Median.

And, even if this reasoning is not accepted, if the asymmetry

[1] See the writer's paper in "Problems in Probabilities," *Phil. Mag.*, Oct. 1886.

of the price-curve should not be regarded as serious, and the central point of the supposed symmetrical complete curve or series be taken as the quæsitum, still, even upon this hypothesis, the Median would have special claims.[1]

Another advantage—or the same otherwise viewed—on the side of the Median is its insensibility to accidental alterations of "weight." You may considerably increase or lighten the weights without causing this Mean to be depressed or elated. In the Arithmetic Mean a large weight happening to concur with an extreme relative price produces a derangement which with reference to the present objective[2] (as distinguished from the "consumption") standard may be regarded as accidental. The Median is free from this fortuitous disturbance. The rationale of this stability is supplied by the Calculus of Probabilities.

It appears, therefore, that our index-number, though not likely to be wide of any mark which has been proposed, is not the one which is most accurately directed to a particular, or rather, indeed, the most general object. It is no matter of surprise or complaint that we should not hit full in the centre an object which has not been our aim; our index-number being mainly a *Standard of Desiderata*, measuring the variation in value of the national consumption. Our primary aim, indeed, is more comprehensive, not this special, but a collective, or "compromise," scope; not so much to hit a particular bird, but so to shoot among the closely clustered covey as to bring down most game. But then we are brought back to, or nearly to, the directer aim and simpler object by a consideration which has great weight in practical economics, the necessity of adopting a principle—as Mill says with respect to convertible currency—"intelligible to the most untaught capacity." Now every tyro in our subject makes straight for the Consumption Standard; but the more delicate distinctions of the Producers' Standard and the typical or quasi-objective index-number evade popular perception.

In view of this practical exigency it may well be that the Committee's index-number is the one best adapted to purposes in general—the principal standard as defined in the First Report.

[1] The problem would then be analogous to the reduction of symmetrical observations relating to a physical quantity. On account of the "discordance" of the price-observations, their very different liability to fluctuation, the writer would recommend the use of the Median on the grounds which he has stated in the paper on "Discordant Observations," *Phil. Mag.*, April 1886.

[2] See the first Memorandum (H), and also *Journal of the Statistical Society*, June 1888.

What is here contended is that, with respect to a certain purpose other than the consumers' interest, the Committee's index-number is on the one hand likely to be a very good measure, and on the other hand not the very best possible.*

B.

We have now to compare index-numbers differing as to the prices operated on as well as the methods of operation. One important case is where the prices of the principal articles are the same for the compared index-numbers, the data differing only as to a small part of the total value. For example, of the total value covered by Mr. Sauerbeck's index-number about $\frac{9}{10}$ is common to the Committee's scheme. For Mr. Sauerbeck's weights (or " nominal values ") of the twenty-one articles common to both calculations make up (for the year 1885) 559, while the sum for all the items treated by him is 617.

Let us see then what difference is caused by operating on all Mr. Sauerbeck's forty-five articles instead of only the twenty-one principal items which are common to his price list and ours. He himself (at p. 595 of the *Journal of the Royal Statistical Society*, 1886) gives us the Means of Comparison :—

	1885.	1873.
Mr. Sauerbeck's Weighted Mean for 45 articles . .	71·2	115·2
The Committee's Weighted Mean for 21 of those articles	70·6	115

It is interesting to observe that the Median does not suffer any change by being extended from twenty-one to forty-five articles. The attention of the reader is invited also to the ease

Above 80.	Between 70 and 80.														Below 80.
Dots to the number of 12.	79	78	77	77	76	75	75	73	73	73	72	72	71	70	Dots to the number of 19.

of this method. In order to take in the twenty-four additional articles we have only to write down a few more figures in the central compartment, to add a few more dots in the extreme compartments, as shown in the annexed diagram. Indeed it is not necessary to record the number of observations (by way of

* Compare the remarks at the end of the first Memorandum above, p. 256.

1	2	3	1 (*continued*).	2 (*continued*).	3 (*continued*).
Relative prices.	Precision.	Another System of Precision.	Relative prices.	Precision.	Another System of Precision.
60	2		72	1	3
62	3	3	73	2	1
63	3	3	73	2	2
64	2	3	73	1	3
59	1	2	75	1	1
65	2	2	75	2	2
58	3	1	76	1	1
64	3	2	77	3	2
50	2	2	77	2	3
55	1	1	79	1	2
55	2	1		1	2
60	2	3		3	1
59	2	3		3	3
57	1	2		2	3
57	1	3		2	3
62	2	2		1	3
63	1	2		3	2
63	2	2		3	2
		1		3	3
70	3	3		1	1
70	2	1		2	1
71	2	2		2	1
72	1	3		1	2
Sums .	43	47	—	43	47

dots) in more than *one* of the extreme compartments. The Median is the *twenty-third* figure in the order of magnitude, that is, 72. Proceeding similarly for the year 1873, we find the Median of Mr. Sauerbeck's forty-five relative prices 109.

Now let us try the effect of weighting. Running my eye over some pages of statistics, I assign the digits 1, 2, 3 as they occur to the comparative prices, which are in pell-mell order up to 70; between 70 and 80 in the order of magnitude; and above 80 are not represented at all. The sum of the whole second column thus formed is 86. The central point corresponding to half that sum is at the foot of the first half of the second column, corresponding to the entry 72 in the first column. Accordingly 72 is the adjusted Median. I try another system of precision-factors arbitrarily assigned. And still the Median is 72!

The comparisons offered by Mr. Sauerbeck's materials are summed up in the table on p. 334.

For estimating the extent of difference to be expected between two index-numbers which overlap as to some of their items, a formula is derivable from the above reasoning. Of course, as the number of items common to two compared index-numbers is diminished the chances of their dissilience are increased. The

	1885.		1873.	
	The 21 articles common to the Committee and Sauerbeck.	Sauerbeck's 45 articles.	The 21 articles common to the Committee and Sauerbeck.	Sauerbeck's 45 articles.
The Simple Arithmetic Mean .	70	74	110·5	111
The Committee's Weighted Mean .	70·6	—	115	—
Sauerbeck's Weighted Mean . .	73	72·5	115	115·2
Jevons' adjusted Geometric Mean .	69	—	—	—
The Simple Median . . .	72	72	108	109
The Median adjusted according to *quantity*	72	—	108	—
The Median adjusted on an arbitrary principle . . .	69·5	72	—	—

art of conjecturing can in such a case throw only a very feeble light on the relation between two such index-numbers. For instance, it could hardly have been predicted that the Simple Arithmetic Mean for Mr. Sauerbeck's forty-five articles should differ so little as ·5 from the same Mean for twenty-one articles, as proved to be the case for the year 1873. It is even more surprising that if for 1885 we complete our index-number, taking account of the six items belonging to our scheme not included by Mr. Sauerbeck, there is a marked rise in the index-number owing to all these six returns being above the average. The following little table is formed by comparing the prices in 1885 with the average for 1866–77 as given in the *Statistical Abstract*:—

Articles omitted hitherto.	Relative prices for 1885.	Weight assigned by the Committee.	Product of Columns 2 and 3.
1	2	3	4
Fish [1]	104	2½	260
Beer	76	9	68
Spirits [2]. . . .	120	2½	300
Wine	100	1	100
Tobacco . . .	85	2½	212½
Caoutchouc . . .	109	1	109
Sums	594	18·5	1665
Means	97	—	90

[1] Fish imported. [2] Spirits other than rum and brandy.

If we add the outcome of this table to that of the first table representing the other twenty-one articles, we have 1665 + 5952 = 7617; which, divided by 100, gives the new index-number 76.

Of course in applying the doctrine of Chances to this problem we must abstract all *animus*. If you pick out the large relative prices and the large weights you will doubtless succeed, like Mr. Forsell, in producing discrepancies—though even his success in that attempt seems less than might have been expected.

In concluding this comparison of results the writer may say, in the phrase of Jevons, that he has taken more than reasonable pains to secure arithmetical accuracy. No doubt mistakes will have come. But, as the calculations have been performed without any conscious bias, it may be hoped that the errors will neutralise each other, and that the general impression left by the work is correct.

APPENDIX.

STATEMENT OF THE EXTENT, AND ESTIMATE OF THE SIGNIFICANCE, OF THE DIFFERENCE BETWEEN THE COMMITTEE'S SCHEME AND OTHERS.

TABLE I.

Articles common to the Committee and Mr. Sauerbeck's Weighted Index-number.	Weights actually assigned.		Weights as Percentages of Total Weight of the Common Articles.		Differences between Columns 4 and 5.
	The Committee.	Mr. Sauerbeck.	The Committee.	Mr. Sauerbeck.	
1	2	3	4	5	6
Wheat	5	61	6	11	5
Barley . . .	5	30	6	5·5	·5
Oats	5	32	6	6	0
Potatoes and rice .	5	32	6	6	0
Meat	10	88	12	15·5	3·5
Butter . . .	7½	23	9	3	6
Sugar	2½	30	3	5·5	2·5
Tea	2½	15	3	2	1
Cotton . . .	2½		3	10	7
Wool	2½	42	3	7·5	4·5
Silk	2½	4	3	1	2
Leather . . .	2½	10	3	2	1
Coal	10	74	12	13	1
Iron	5	27	6	5	1
Copper . . .	2½	7	3	1	2
Lead	2½	3	3	·5	2·5
Timber . . .	3	17	4	2	2
Petroleum . . .	1	3	1	·5	·5
Indigo . . .	1	3	1	·5	·5
Flax	3	4·5	4	1	3
Palm oil . . .	1	1·5	1	0	1
Sums . . .	81·5	564	98	98·5	46·5

TABLE II.

Articles common to the Committee and Mr. Palgrave.	Weights as Percentages of Total Weight of the Common Articles.		Differences between Columns 2 and 3.
	The Committee.	Mr. Palgrave.	
1	2	3	4
Wheat	10	19	9
Meat	20	25	5
Sugar	5	7	2
Tea	5	3·5	1·5
Tobacco	5	1	4
Cotton	5	12	7
Wool	5	6·5	1·5
Silk	5	·5	4·5
Leather	5	3·5	1·5
Iron	10	7	3
Copper	5	1·5	3·5
Lead	5	·5	4·5
Timber	6	7·5	1·5
Indigo	2	0	2
Flax	6	2	4
Oil [1]	2	2	0
Sums	101	98·5	54·5

[1] *Palm oil* in the Committee's scheme; *oils* in Mr. Palgrave's.

TABLE III.

Articles common to the Committee and Mr. Sauerbeck's Unweighted Index-number.	Weights actually assigned.		Weights as Percentages of Total Weight of the Common Articles.		Differences between Columns 4 and 5.
	The Committee.	Mr. Sauerbeck.	The Committee.	Mr. Sauerbeck.	
1	2	3	4	5	6
Wheat	5	3	6	8·5	2·5
Barley	5	1	6	3	3
Oats	5	1	6	3	3
Potatoes and rice	5	2	6	6	0
Meat	10	6	12·5	17	4·5
Butter	7½	1	9	3	6
Sugar	2½	2	3	6	3
Tea	2½	1	3	3	0
Cotton	2½	2	3	6	3
Wool	2½	2	3	6	3
Silk	2½	1	3	3	0
Leather	2½	2	3	6	3
Coal	10	2	12·5	6	6·5
Iron	5	2	6	6	0
Copper	2½	1	3	3	0
Lead	2½	1	3	3	0
Timber	3	1	4	3	1
Petroleum	1	1	1	3	2
Flax	3	1	4	3	1
Indigo	1	1	1	3	2
Palm oil	1	1	1	3	2
Sums	81·5	35	99	103·5	45·5

TABLE IV.

Articles common to the Committee and Dr. Soetbeer.	Weights actually assigned.		Weights as Percentages of Total Weight of the Common Articles.		Differences between Columns 2 and 5.
	The Committee.	Soetbeer.	The Committee.	Soetbeer.	
1	2	3	4	5	6
Wheat	5	2	Practically same as Column 2.	4·5	·5
Barley	5	2		4·5	·5
Oats	5	1		2	3
Potatoes and rice	5	2		4·5	·5
Meat	10	4		9	1
Fish	2½	2		4·5	2
Butter, milk, and cheese	7½	2		4·5	3
Sugar	2½	2		4·5	2
Tea	2½	1		2	·5
Beer	9	1		2	7
Spirits	2½	3		7	4·5
Wine	1[1]	2		4·5	3·5
Tobacco	2½	1		2	·5
Cotton	2½	1		2	·5
Wool	2½	1		2	·5
Silk	2½	1		2	·5
Leather, etc.	2½	3		7	4·5
Coal	10	1		2	8
Iron	5	3		7	2
Copper	2½	1		2	·5
Lead	2½	1		2	·5
Timber	3	3		7	4
Indigo	1	1		2	1
Flax	3	1		2	1
Palm oil	1	1		2	1
Sums	98	43	98	94·5	52·5

[1] *Hops.*

TABLE V.

Articles common to the Committee and Jevons.	Weights actually assigned.			Weights relative to the Total Weight of the Common Articles.			Differences between Columns 6 and 5.	Differences between Columns 7 and 5.
	The Committee.	Jevons.		The Committee.	Jevons.			
		a[1]	b[2]		a[1]	b[2]	a[1]	b[2]
1	2	3	4	5	6	7	8	9
Wheat . .	5	1	1	7	3·5	2	3·5	5
Barley . .	5	1	1	7	3·5	2	3·5	5
Oats . .	5	1	1	7	3·5	2	3·5	5
Meat . .	10	3	5	14·5	11	9	3·5	5·5
Butter and cheese	7½	1	3	11	3·5	6	7·5	5
Sugar . .	2½	1	3	4	3·5	5	·5	1
Tea . .	2½	1	4	4	3·5	7	·5	3
Spirits . .	2½	1	3	4	3·5	6	·5	2
Cotton . .	2½	3	3	4	11	6	7	2
Wool . .	2½	1	2	4	3·5	4	·5	0
Silk . .	2½	1	3	4	3·5	6	·5	2
Leather . .	2½	2	4	4	7	7	3	3
Iron . .	5	3	3	7	11	6	4	1
Copper . .	2½	1	1	4	3·5	2	·5	2
Lead . .	2½	1	4	4	3·5	7	·5	3
Timber . .	3	2	6	4·5	7	11	2·5	6·5
Flax . .	3	1	1	4·5	3·5	2	1	2·5
Indigo . .	1	1	1	1·5	3·5	2	2	·5
Palm oil . .	1	1	1	1·5	3·5	2	2	·5
(Wine) . .	(2½)	—	4	(4)	—	7	—	3
Sums .	68	27	54	101·5	96	101	45·5	57·5
	(70·5)	—	—	(105·5)	—	—	—	—

[1] First form of index-number based upon 39 articles ("Serious Fall").
[2] Second form of index-number based upon 118 articles (*ibid.*).

TABLE VI.

Index-numbers compared with the Committee's.	Mr. Sauerbeck's Weighted.	Mr. Palgrave's.	Mr. Sauerbeck's Unweighted.	Dr. Soetbeer's.	Jevons.	
					a	*b*
Number of articles common to the Committee's and other index-numbers	21	16	21	25	19	20
Mean difference (per cent.) between the weights of the common articles according to the Committee's and other schemes	50	54	45	53	45	58
Weight of the common articles per cent. of the weight of all the articles in the Committee's scheme	81·5	50·5	81·5	98	68	70·5
Weight of the common articles per cent. of the weight of all the articles in other schemes	90·5	98	78	44	70	54
Discrepancy as likely as not to occur between the Committee's and other results	2	2·5	2	2	2·5	2·5
Discrepancy very unlikely to occur between the Committee's and other results	8	11	8	8	10	11

Remarks upon the preceding Tables.

These tables present a comparison between the index-number proposed by the Committee and some other well-known constructions of the same kind. In the first five tables the feature of comparison consists of those articles or items which are common to the Committee's and the compared schemes. The tables show the different importance or "weight" assigned to the same items in the Committee's and each of the other schemes. For the purpose of exhibiting this difference it is proper to contrast, not the actual weights employed by the Committee and each compared index-number, but the weights relative to the total weight assigned to the common items by the Committee's and the compared scheme respectively. Thus, in the first table, the first column states the articles, twenty-one in number, which are common to the Committee's index-number and to one which has been given by Mr. Sauerbeck (*Journal Statistical Society*, 1886, p. 595). The second and third columns give the weights actually affixed by the Committee and Mr. Sauerbeck respectively to the comparative prices of those twenty-one articles. The third and fourth columns give the weights relative to the total weight of the coincident portions of the two systems. Thus, 61 being the weight actually assigned by Mr. Sauerbeck to wheat, while 564 is the sum of the weights attached by him to all the articles common to him and the Committee, $\frac{61}{564}$, or the same fraction multiplied by 100 (= 11 nearly), is taken as the proper weight according to Mr. Sauerbeck for wheat; in a curtailed index-number covering only those articles common to him and the Committee. By parity $\frac{5}{81 \cdot 6} \times 100$, or six nearly, is the weight for the same article according to the Committee. In the sixth column the differences—the absolute differences without regard to *sign*—between the respective weights are given. To appreciate the importance of this difference of weight, we must consider it in relation to the absolute (mean) weight. Thus $\frac{\text{Mean difference of weight}}{\text{Mean weight}}$ is the fraction (or, multiplied by 100, the percentage) which most, or at least very, properly measures the discrepancy between the two systems. Now the Mean weight for each of the two compared systems is $\frac{100}{21}$. Therefore we have for the required measure

$$\frac{\text{Sum of differences}}{21} \div \frac{100}{21}, \text{ or simply } \frac{\text{Sum of differences}}{100}$$

(or, expressed as a percentage, the sum of differences). Thus in the case before us the average deviation between the compared weights is 49·5, or 50 per cent. (nearly). This figure is useful as enabling us (taking into account the number of common items) to predict the extent of discrepancy which is likely to exist between the results of the two methods of treating the common data.

The second table presents a similar comparison between the Committee's and Mr. Palgrave's index-number (" Third Report of the Committee on Depression of Trade," Appendix B). It has not been thought necessary to record the actual weights. Those employed in the computation of the " relative " weights according to Mr. Palgrave were the figures of *comparative importance* given by him for the year 1885, which differ very little from the corresponding entries in previous years. The coefficient of discrepancy between the two results being much the same as in the former comparison, we may expect much the same difference, or rather one somewhat larger, since the number of common items (sixteen) is here somewhat smaller (than twenty-one).

The remaining index-numbers do not equally admit of being laid alongside that of the Committee for the purpose of comparison. They are as it were in a different plane, adopting a different formula (as well as different constants) from the Committee. In these schemes, unlike the Committee's, each comparative price is not affected with a factor or weight corresponding to its importance. *Prima facie* every relative price counts for one; but the principle of weight is to some extent asserted by introducing as independent items several species belonging to one genus. Thus in Mr. Sauerbeck's *unweighted* index-number, our Table 3, there figure *two* species of *wheat* and also one of *flour;* in effect assigning a weight of *three* to *wheat.* There is indeed something arbitrary in such interpretation. For in comparing this sort of index-numbers with the Committee's it is hardly possible—as in the case of the explicitly weighted index-numbers—to suppose the prices (for the common articles) to be the *same* in the two compared calculations. For example, our price of wheat is taken from the "Gazette"; theirs may be a Mean of that price and the price of flour. Accordingly the estimate of the difference to be expected (proportioned to the total of the last column) is apt to be less accurate, to be under the mark, in these cases. A further inaccuracy affects this estimate in the case of Jevons' index-number, our Table 5, namely, that he adopted the *Geometrical* method of combining relative prices. In fact, our estimates apply only to the *Arithmetic* combination of Jevons' materials, to be supplemented by the observed fact that the Arithmetic and Geometric Means of prices do not much differ.

The last table resumes the results of the first five in its first and second rows. The first row states the number of items common to the Committee with each of the compared schemes—a necessary datum for the estimate of the discrepancy likely to exist between the results. *Ceteris paribus,* this discrepancy is *inversely* proportioned to the *square root* of the number of common items. The second row gives the mean difference between the respective weights as above defined. The third and fourth rows compare the Committee's index-number with each of the others as to the extent of the materials not common to both. The comparison may be thus illustrated. Let CO represent by its length the

quantity of weight common to the Committee and the other index-number. Let CC′ represent the total weight of all the articles in the Committee's system, and OO′ that of the other system. The third row gives the ratio of CO to CC′, and the fourth column the ratio of CO to OO′.

The last two rows give an estimate of the discrepancy likely or unlikely to occur between the results of the compared computations. This estimate involves (in addition to the data contained in the preceding rows) a constant or coefficient deduced from the course of English prices in past years : the inequality or *dispersion* of price-variations, which keeps pretty constant from year to year. The estimates are therefore only applicable to England. They are to be taken *cum grano*, with the reservations stated in various parts of the Memorandum.