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ON THE DETERMINATENESS OF ECONOMIC EQUILIBRIUM

[This is a translation of an article which appeared under a slightly different title in the *Giornale degli Economisti*, 1891. The inquiry takes its start from a passage in the then recently published second edition of Marshall's *Principles*, where he adduces from the present writer's essay on *Mathematical Psychics* a construction there largely employed in the investigation of economic equilibrium, the *contract-curve*. Apart from this connection the discourse is not closely related to the text. For Marshall in the passage cited has in view a market in a special sense distinguished from normal, whereas the process which I analyse has much in common with the determination of normal equilibrium. Besides, as argued by Mr. Berry in the same volume (XL.) of the *Giornale*, the term " determinate " is used by Marshall in a somewhat different sense from that which I have adopted.

Apropos, it may be remarked that there is a certain indeterminateness about the use of the term " determinate " by economists. Thus Pareto has demurred to the description (above, **E**) of the transactions between two monopolists as indeterminate. There being more equations than unknowns, the problem, he thinks, cannot properly be described as " mathematically indeterminate."

I dispute no man's definition of terms; concerned, rather, with the truth of propositions. The proposition which I seek to establish here relates to a typical market consisting of two groups of individuals, say A's and B's : the A's offering the commodity *a* in exchange for β supplied by B's. Each A makes agreements with B's independently of and not in concert with other A's; and the B's likewise act independently. The term " market " applied to this transaction is not to be understood in a sense opposed to " normal " or " natural." Rather there is conceived to be a certain normality about the proceedings. They need not be supposed to take up a long period; rather the contrary, since the disposition and circumstances of the parties are assumed to remain throughout constant. But it is supposed

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that agreements are renewed or varied many times. A " final
settlement " is not reached until the market has hit upon a set
of agreements which cannot be varied with advantage to all the
re-contracting parties. The re-contract most favourable to the
disturbance of a temporary equilibrium is one in which an A deals
with a great many B's. If that power is not used, e.g., if each
A confines himself to dealing with one B, it is quite possible (as
will appear below) that re-contract thus hampered would not
disturb the equilibrium. Thus the condition that perfect finality
should be reached may be stated as follows, in the case, say, of
equal numbers on both sides of the market, m A's and m B's;
it must be impossible for any number of A's, say m—n, dealing
(each for himself) with any number of B's to enter into a new set
of agreements with advantage to all the re-contracting parties.
Since in general the less restriction there is on the number of B's
who re-contract the greater is the possibility of a new equilibrium,
the condition is adequately expressed by the proviso that it
should not be possible for (m—n) A's to re-contract with all
the B's. This " all " has proved a stumbling-block to a critic
who writes in the same Giornale for June, 1891. But I think
it might have been difficult to recall the explanation above given
by any other concise phrase. I trust that I shall not suffer like
the plaintiff in the old state of the English Law who lost his case
because in describing an article which had been stolen from him
he spoke of a " ham " where he should have used the words
" part of a ham." At any rate before condemnation is passed,
reference should be made to the writer's essay on Mathematical
Psychics ; on which the article in the Giornale is largely based.
In the reproduction of the article here presented I have omitted
several long passages which purported to be restatements of
theories more accurately enunciated in that essay.

I illustrate the theory of determinate equilibrium by two
examples in which first appearances are deceptive. There is first
the case in which the marginal utility of both articles for at least
one of the parties varies with the terms of an agreement. This
circumstance may seem at first sight unfavourable to determinate-
ness. It may be to some kinds of determinateness, but not to
that which has been above defined. Rather, it is the general
rule that both articles should vary in respect of final utility for
both parties. If we have to do not with the general problem,
but with the particular case in which the marginal utility of
one commodity remains constant—as in the instance cited below
from Auspitz and Lieben—the equilibrium does not on that

account, I think, become more determinate than in the general
case, theoretically at least, and apart from " friction."

If this view is correct, indeterminateness is not to be attributed
to the labour market, because the marginal utility of money
varies with the price of labour that may be set up. But a certain
indeterminateness is to be attributed to that market for a quite
different reason, namely, the circumstance that a man cannot,
or at least does not, simultaneously serve two masters. This
point is disputed by the aforesaid critic in the *Giornale* (June,
1891). But, as appears from the passages cited below from the
Giornale, he seems not to have taken account of an essential
condition in our problem, viz., that the competing work-people
should not act in concert.]

The theory of Exchange is founded on the principle of Barter,
which has been discussed by Marshall with remarkable originality
and accuracy. He has avoided the common error of attributing
to two persons who are bargaining with each other a fixed rate
of exchange governing the whole transaction. A uniform rate
of exchange, he remarks, is applicable only to the case of a perfect
market. By way of example he puts the case of A having a
basket of apples and B a basket of nuts; A desiring nuts and B
apples. Referring to this example, I would express the process
of barter mathematically in the following manner.

Let the abscissa x denote the number of apples given by A
and received by B; the ordinate y, the number of nuts given by B
and received by A. Thus every point in the plane (x, y) repre-
sents a barter of so many apples for so many nuts. Let u be
the utility, or satisfaction, of A so far as it depends on the one
hand on the number of nuts that he gains, and on the other hand
the number of apples that he retains, that is the number initially
in the basket less by the number that he has parted with. Let
v be the similarly defined advantage of B.[1] Bartering will
continue as long as it is possible for both parties to gain thereby.
Let Δx be the *quid* given by A and Δy the *pro quo* received by
him at any stage of the transaction. The process of exchange

[1] It may assist the formation of correct conceptions to put $u \equiv \phi_1(a - x) + \psi_1(y)$, where a is the original stock of apples; with a corresponding expression for v. But we are not limited to this simple form. We are at liberty to use for it an expression of the form $\chi(a-x, y)$.

can only continue as long as the gain (of satisfaction) by A and likewise B's gain is positive; in symbols,

$$\frac{du}{dx}\Delta x + \frac{du}{dy}\Delta y > 0. \quad \frac{dv}{dx}\Delta x + \frac{dv}{dy}\Delta y > 0.$$

Now this condition will cease to be fulfilled when the total quantities exchanged, x and y, are such as to satisfy the equation

$$\frac{du}{dx}\frac{dv}{dy} = \frac{dv}{dx}\frac{du}{dy}.$$

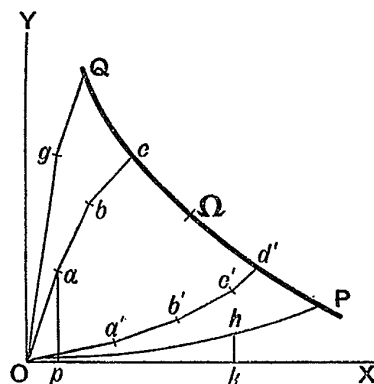The locus thus represented I have called the *contract-curve*.

Fig. 1.

In Fig. 1 any point, for instance a, denotes the exchange of, or contract for the exchange of, the number of nuts represented by the ordinate ap against the number of apples expressed by the abscissa Op. The series of short lines Oa, ab, bc, corresponds to successive barters (at different rates of exchange) of a few nuts for a few apples. The broken line $Oa'b'c'd'$ indicates a possible set of exchanges more favourable to B.

Alike at c and at d' the bartering comes to a stop, those points being situated on the contract-curve PQ. Of this curve the only part with which we are concerned is that which is intercepted by the *curves of constant satisfaction*, or *curves of indifference*, for A and B respectively which pass through O; viz., OhP and OgQ. The *curve of indifference* which passes through a given point is the locus of all the contracts that procure to the con-

tractor the same satisfaction as the contract designated by the given point. For instance, the contract designated by the point h, that is Ok apples given for hk nuts, or, again, the contract designated by P, procures for the contractor A the same satisfaction as the contract (or absence thereof) designated by O. It is indifferent to A whether he makes either of the first two deals, or none at all. Similarly, it is indifferent to B whether the dealing is represented by the point O, or g, or Q.

At what point on the tract of contract-curve between P and Q the process of bartering will come to a stop cannot be predicted. The position of equilibrium may be described as indeterminate. The essential condition of this indeterminateness is the absence of competition.

The essential condition is not to be sought in an incident of the case before us, namely, that the marginal utility of both the commodities varies in the course of the dealing. The phenomenon of indeterminateness may very well exist without that incident. Whether or not the marginal utility of one commodity, say y, is regarded as varying with the additions or subtractions incidental to exchange, there will always remain—in the case of barter between two individuals—an indeterminate tract on the contract-curve, every point of which is a position of equilibrium. True, the curve will sometimes degenerate into a right line parallel to one of the axes.[1] An example is furnished by Messrs. Auspitz and Lieben in that part of their important book in which they discuss the contract between a monopolist entrepreneur and a union of operatives. Assuming that the marginal utility of money may be treated as constant, they justly observe that "the determination of price seems to be between wide limits arbitrary."

Thus the imperfections of the labour market do not depend on the circumstance that the marginal utility of money varies for the work-people according to their bargain with the employers, according as their wages afford only bare necessities or superabundant luxuries. The imperfections of that market are rather

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[1] In symbols if the final utility of y is constant for both dealers, we may put for u the satisfaction of the contractor A the expression $\phi_1(a-x) + ay$, and for v (pertaining to B) $\phi_2(x) + \beta y$; where a and β are constants. Accordingly we obtain for the contract-curve $\dfrac{\phi_1'(a-x)}{a} = \dfrac{\phi_2'(x)}{\beta}$; or $x =$ constant; representing a line parallel to the axis of y. This line fulfils the characteristic condition of the contract-curve that at any point of the locus the inclination of the tangent to a curve of indifference passing through that point should be the same for both parties.

to be sought in certain peculiarities which Marshall has pointed out, as noticed at the conclusion of this paper.

Apropos of impediments to the play of competition in the labour market there may be observed from the point of view here adopted two incidents which are more curious than important.

Suppose that every A can contract with only one B, and likewise every B with only one A. Then it is no longer possible that $(m-n)$ A's—each acting independently—should form a set of new contracts with all—or any number of—the B's (and likewise impossible for $(m-n)$ B's to re-contract); which variation of contracts—when it can be effected with advantage to all the re-contracting parties—is here regarded as the essential attribute of competition. Accordingly, the equilibrium would be in the case supposed as indeterminate for a set of couples, as we have seen that it is in the case of a single couple. There may possibly exist types of domestic service which fulfil the supposed condition. But practical importance is not claimed for this *curiosum*.

There is another unobserved peculiarity of the labour-market which is the more curious in that it constitutes a positive advantage to the work-people in their dealings with entrepreneurs. Suppose that the system of contracts is initially at any point on the contract-curve (Fig. 1) d' on the right of the position hereinafter defined Ω; that is to the advantage of the B's who supply the article y. It may be shown that the advantage which the B's thus possess is lost through the action of $(m-n)$ of their number who carry off, so to speak, the whole (or a large part) of the A group.* Suppose now that the A's are work-people, the B's entrepreneurs. The terms first proposed may be very much to the advantage of the entrepreneurs. But they will lose that advantage through competition against each other. Assuming that an entrepreneur can employ several men, it will be to his advantage to offer to some of his rivals' workmen better terms than they were receiving, and so carry them off. And this process will theoretically continue until the system of agreements reaches the position of stable equilibrium symbolised by the point Ω, the point at which the demand curves (not shown in the figure) intersect on the contract-curve. Analogously it might be supposed that if the terms are at first too favourable for the work-people, if, for instance, the point c (left of Ω) represented an initial system of contracts, the work-people would lose that advantage, by mutual competition. But such competition would imply, according to our analysis, that a workman

* See note at end of article.

takes on several entrepreneurs; that a man *can* serve two masters
simultaneously. In the case of painters, no doubt, and many
home-workers plurality of employers is common. Still, the
general rule is that no man can serve two masters, and so far as
this is true the work-people have an advantage over the entre-
preneurs in that they cannot equally beat down the price of
their services by mutual competition.

I do not, however, regard these nice points as more than
curiosa, of little practical importance in comparison with the
conditions of the labour-market on which Marshall has dwelt;
in particular, the tendency of any accidental disadvantage under
which the work-people may be suffering to become perpetuated
through the lowering of their vitality and efficiency, and the fact
that employers are few in comparison with the number of work-
people. In this field Marshall has thoroughly reaped the harvest,
leaving nothing to those who come after him but to glean some
logical niceties.

(Note referring to p. 318.)

[The argument that if a rate of exchange unduly favourable to the
work-people is set up they will not beat the price down by their mutual
competition is disputed by an able critic in the June number (1891)
of the *Giornale*, on grounds of which the following quotation (*loc. cit.*
p. 553) contains the gist. "Though it is true that 'no man can serve
two masters,' yet it is quite possible for a number of work-people to
increase the number of entrepreneurs whom they serve. If, on
average, 100 work-people serve one employer, 500 may at pleasure
renew their contract with six or with four. The first proceeding
tends, according to Professor Edgeworth's reasoning, to favour the
entrepreneurs, the second the work-people."

To which I reply in the October number of the *Giornale* (1891)
suggesting that the critic has not taken account of the condition that
the work-people should not act in concert.

"Say five entrepreneurs employ each on average 100 men. Four of
these entrepreneurs are disposed to employ a larger number of work-
men, and they offer higher wages, each finding that he can thus make
a bargain more advantageous for himself and for his employees. What
can be simpler? But now consider the opposite case. How can 400
work-people originally employed by four entrepreneurs find occupation
with five entrepreneurs by way of an initiative on the part of the
work-people, each acting independently, and not in concert with
others. This case is not analogous to the first; because it is not in
the power of any operative to purchase, so to speak, a fraction of an
entrepreneur, other than, or in addition to, that which he already
enjoys in virtue of his present engagement."]