(e)

APPLICATION OF THE DIFFERENTIAL CALCULUS TO ECONOMICS

[THIS fragment, published in *Scientia*, 1909, under a slightly different title, deals with the mechanics of wealth in the narrow sense which excludes what Jevons calls a " common denominator of feeling." The principle of " recontract " (above, β) is employed in a variant proof of the theorem that the position of stable equilibrium is at the intersection of the demand-curves pertaining to the respective sides of a market. The abstract theory of Supply-and-Demand is applied to international trade; and to the analogous case of transactions between the parties to Distribution. But it is pointed out that this analogy may prove misleading if there is not taken into account a certain characteristic of domestic trade which Professor Pigou has described as the elasticity of the Demand for Common Labour (cp. *Economics of Welfare*, Book V. ch. iii. s. 8). The conceptions of Producers' Surplus and of the margin of production are illustrated by critical references to well-known writers.]

Description and Division.

The extension to Social Science of principles approved in Mathematical Physics is a theme worthy of *Scientia*. For this journal, as announced at its inception, is " born of the desire to co-ordinate the work carried on in different fields of knowledge." Alone in a world of specialists it seems to realise that part of Plato's scheme of education which consisted in bringing the sciences together and contemplating them in their mutual relations.

There is indeed a certain resemblance between the ancient philosophy and the modern study of Mathematical Psychics in so far as both attain large general views rather than particulars adapted to art and practice. But we are not committed to the contempt of fact which seems to characterise the Platonic precept that " in astronomy, as in geometry, we should employ

problems, and let the heavens alone, if we would approach the subject in the right way." [1] The Newtonian astronomy is rather the model of our Science; but we can only follow it at a great distance owing to the multiplicity of variables in Social Science and the want of a unit for measuring advantage in a subjective sense. Often we must be content with knowing that knowledge is unattainable without more data than we possess—the Socratic lesson of modesty which was taught by Cournot and Jevons.[2]

This description is applicable generally to the subject designated by the main portion of my title—" the use of the Differential Calculus "—and more particularly to that part of the subject which is demarcated by the limiting clause—" to determine conditions of maximum advantage." The subject thus limited is, if we understand " Economics " in its largest sense, almost coincident with—only slightly narrower than—the field which is covered by the well-known dictum of Malthus : " Many of the questions, both in morals and politics, seem to be of the nature of the problems *de maximis et minimis* in fluxions." Such is the little territory, on the borderland between Physics and Psychology, which I attempt here to survey.

One main division has already come into view, that which separates Economics in a narrow or " proper " sense from the wider Science of Political or rather Social Economy, of which the object is not simply wealth, but welfare, so far as dependent on wealth. We might define Economics proper—the subject of our Section I—by the limitation that in Jevons's words " the motive in one mind is weighed only against other motives in the same mind, never against the motives in other minds." [3] Whereas in Social Economy—the subject of Section II *—what Jevons calls " a common denominator of feeling " must be postulated.

According to the postulate, the relation between the individuals of a Society may be described by the familiar metaphor of

[1] *Republic*, vii. 529, Jowett's translation; a precept which Macaulay contrasts unfavourably with Bacon's method.

[2] The use of Mathematics to make clear the nature and extent of the assumptions implied in dealing with economic problems has been noticed by Professor J. S. Nicholson in a recent lecture (*Transactions of the Faculty of Actuaries*, No. 35, Vol. I. Part IV.).

[3] *Theory of Political Economy*, edition 3, p. 14. *Cp.* Pareto, *Manuel d'Économie Politique*, chap. iii. § 11.

* The design of a sequel dealing with a subject wider than economics proper, was not destined to be executed. The definition of the wider subject may be illustrated by the writer's article in the ECONOMIC JOURNAL for December 1923.

solidarity. But the physical analogy of Economics proper does not lie so ready to hand. The particles of an economic system neither cohere as a solid, nor collide with the independence of a gas. Their liquid movements are comparable to a dance in which youths and maidens move in unison; harmoniously, but subject to a change of partners.

To learn the steps of this peculiar dance consider first the movement of a single couple. One of the pair, say X, gives to the other, say Y, a quantity x of the commodity x in return for the amount y of the commodity y given by Y to X. The terms of such an arrangement may be represented by a point (x, y) on the plane of the paper. The variation of the terms of contract is represented by the movement of the point in the plane. The parties cannot move separately. They are in the same boat. They both have hold of the rudder, but the directions in which they respectively prefer to move are not the same. The magnetic pole towards which X would steer lies to the south-east, supposing that the axis of y points to north. The centre of attraction for Y lies to the north-west. Motion is possible only as long as both parties are winning towards their respective goals.[1]

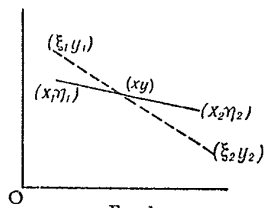

FIG. 1.

Next let there be two individuals on each side, say X_1 and X_2 dealing with Y_1 and Y_2. (The motives or dispositions of X_1 may at first be supposed identical with those of X_2; the psychic forces acting on Y_1 and Y_2 being also identical.) X_1 is free to move in any direction to any distance, provided that either Y_1 or Y_2 accompany, or at least move parallel to him; and the like is true of X_2 with regard to the Ys, and of each of the Ys with regard to the Xs. The conditions are expressed by Fig. 1, in which the point (x_1, η_1) represents the amount of commodity x which the individual X_1 gives and the amount of commodity y which he receives (partly, it may be, from Y_1, partly from Y_2). The point (ξ_1, y_1) denotes the amount of commodity y which Y_1 gives and the amount of commodity x which he receives (in general, partly from X_1, and partly from X_2). The points (x_2, η_2), (ξ_2, y_2) are similarly interpreted. The two points (x_1, η_1) and (x_2, η_2), may be considered as moving along a rod which passes

[1] See the construction in the writer's *Mathematical Psychics*, Part II. p. 20 *et seq.*, and *cp.* Pareto, *Manuel d'Économie Politique*, ch. iii. § 58 *et passim*.

through a point (x, y); subject to the condition that those two points are equi-distant from the point (x, y). The points (ξ_1, y_1) and (ξ_2, y_2) are similarly related to the point (x, y); which may thus be described as the centre of gravity of the system. The system moves towards a position of equilibrium—a position of maximum energy, as it may be considered—under the influence of forces acting on the constituent particles; subject to the constraints which have been indicated. Among those conditions is one very unusual in Physics, very difficult to represent mechanically, that no part of the system does work against another. For instance, the point (x_1, η_1) cannot suffer a change by the subtraction of Δx_1 from x_1, the addition of $\Delta\eta_1$ to η_1, if such a step in the direction defined by the ratio $\Delta\eta_1 : \Delta x_1$ is one in which the individual is averse to move. The conception may be extended to any number of Xs on one side, and any number of Ys on the other. The forces actuating the different members of one group need not be identical; they need to act, not in exactly the same *direction*, but in what may be called the same *sense*.

As to the variation of the forces acting on a particle with the variation of its position—the change in the motives of an individual, *e.g.*, X_r (or Y_r) with the amount of x (or y) that he retains, and the amount of y (or x) that he has obtained—the natural and usual supposition is that the forces are a function only of the position defined by those amounts, viz. x_r and y_r. This state of things is here designated by the term "independent dealing," the symbol (A). But we shall also have to entertain the supposition, not very usual in Physics, that the forces acting on any particle at the point (x, y) depend not only on the co-ordinates which define that position, but also on the position of the system—upon the co-ordinates of its centre of gravity, as we may say, when the number of particles on each side is equal. The supposition will be more fully explained in the sequel, under the head of "interdependent dealing," labelled (B).

I do not attempt here to cultivate the fields which have been indicated; but as I pass in the course of a rapid survey, I may sometimes root up a weed which has proved noxious, or drop a seed which may germinate.

I. Economics Proper

(A) *Independent Dealing.*

Simple Exchange.—This heading is meant to designate the simplest form of *market*, the conception of which has been attained

in the introductory description of two groups of Xs and Ys, dealing respectively in two commodities x and y. This is the economic molecule; itself, as we have seen, a compound of curiously interrelated atoms. In the fragmentary system of only two particles, their identical position (which may be equally in our notation denoted by the symbols (x, η), (ξ, y), (x, y)) would be varied under the influence of forces tending to maximum advantage up to a limit investigated by Jevons, in his *Theory of Exchange*, and now commonly known as the *Contract-Curve*.[1] When there are one, two or more dealers on each side of the market, a like condition of contract must still be fulfilled. Thus in the case above put of two dealers on each side, either of the Xs, say X_1, will continue to deal with one of the Ys, say Y_1, up to a point beyond which it is impossible to advance a single step with benefit to both parties. The like is true of X_1 with respect to Y_2, and of X_2 with respect to either of the Ys.

Moreover, in addition to this purely *contractual* condition, the circumstance of *competition* introduced by the duality of the dealers on each side imports a new condition. There cannot be equilibrium unless the slope of the final step taken by any X in conjunction with (parallel to) any Y is the same as the slope of the final step taken by (that or) any other X with any other (or that) Y. For if it be possible let Fig. 1 correspond to a state of equilibrium in which X_1 gives to Y_2 a final increment of x, say $(\Delta x_1)_2$, in return for the final increment of y $(\Delta y_2)_1$; while X_2 gives to Y_1 the final increment $(\Delta x_1)_1$ and receives the final increment $(\Delta y_1)_2$; this arrangement cannot stand if the slope designated by the respective pairs of corresponding increments are different. For in that case it would in general be to the advantage of one of the Xs and one of the Ys to desert their respective partners and take one step at least, and probably several, with each other.

Yet another condition is imposed by competition when we advance to the typical case of indefinitely numerous dealers on each side of the market—a crowd of competitors all of the same order in respect of the possible magnitude of their transactions.[2] Suppose that, in accordance with the conditions of equilibrium which have been already established, all the economic

[1] See the writer's *Mathematical Psychics*, p. 21 *et seq.*

[2] Or more generally of such relative magnitude that the power of any one competitor to influence the market is very small. This condition may be illustrated by its analogy with the condition which must in general be fulfilled by the relative magnitudes of independent statistical quantities, in order that their aggregate may fluctuate in accordance with the normal law of error.

particles are disposed at points such that no *indefinitely small* change of the system is possible; the final step which any one X can be induced to make in conjunction with one Y having the same slope for all, corresponding to a rate of exchange for small quantities $\Delta\eta : \Delta x$. Now by the assumption introduced in this paragraph, it is open to any individual on either side of the market, say X_r (or, *mutatis mutandis*, Y_r), to supply himself by exchanging small quantities with a great number of dealers. In other words, his whole course from zero to the point (x_r, η_r) may be made up of steps taken in conjunction with different Ys, each small step in the direction of the common slope $\Delta\eta : \Delta x$. If now X_r by multiplying steps of this kind can reach a point, say (x'_r, η'_r), which represents greater advantage to him than the point at which he was just now supposed to be at (x_r, η_r), he will tend to proceed to that point. He cannot do so, it may seem, because there will not be increments of y forthcoming on the terms offered to proceed to the new point; since they are as advantageously employed by their owners in dealing with other Xs. But it will be worth the while of X_r to offer rather better terms (with respect to short terminal steps) to a number of Ys than those which are represented by the given slope; since if it is advantageous for X_r to move from (x'_r, η_r) to $x'_r, \eta'_r)$, it will in general [1] be advantageous for him to move to some point in the neighbourhood of (x'_r, η'_r). Thus the system cannot be in equilibrium unless any individual dealer X_r (and similarly each Y) whose dealing consists of η_r received in return for x_r given, obtains as much y as he is willing to take at the rate of exchange defined by the ratio $\eta_r : x_r$. In other words, the point (x_r, η_r) is on the *demand-curve* of the individual X_r, and accordingly the point of equilibrium for the system is on the collective demand curve which represents the total demand (on the part of the Xs) for y at each compared rate of exchange between x and y. Likewise the point of equilibrium is on the collective demand-curve which represents the total demand on the part of the Ys for x; or, in other words, the curve representing the supply of y. Therefore the point of equilibrium is at an intersection of the collective Supply and Demand Curves.[2]

It may be asked, what is the good of thus deducing an obvious matter of fact, that there is one price in a market? I reply,

[1] In the absence of *singularities*, the usual continuity in the functions with which we are concerned may postulate.

[2] For a variant deduction of the familiar generalisation, see *Mathematical Physics*, p. 34 *et seq.*

may we not hope for the sort of advantages which explanation of empirical laws affords in Physics—always enlarged breadth of view, often increased power? We should hope humbly, indeed, mindful of the imperfections of the human, as compared with the physical, sciences.

The hope may derive some encouragement from the instance which comes nearest to our case, being the explanation of a law which applies to human affairs, in part at least, though its principal applications are physical. I refer to the Law of Error, the so-called Gaussian law, which in the present connection at least might more properly be called Laplace's law;[1] the law prevalent in Statistics, which is explained by the action of numerous independent agencies supposed to underlie statistical phenomena.

It may be observed, parenthetically, that there is a resemblance not only between the subject-matter of the two laws explained—so far as both relate to human affairs—but also in the causes which form the explanation, the action of great numbers being a necessary part of each cause. But the relation of the economic to the statistical theory is rather a resemblance than an analogy. There is not one law, but two laws; the law of competition, and the law of averages. They are to be condemned who say or imply that the steadiness of normal value is altogether explained by the principle of Statistical Stability.[2]

To return to our explanation of a market; it may be held that this shows badly compared with the explanation of the Law of Error and other more familiar derivations of observed uniformities. For there is no other probable explanation available in many of those cases; whereas here there is a very simple alternative explanation, namely, that exchange with different parties at the same rate is dictated by the simplicity of the

[1] For a summary of the present writer's views on this law see the paper on "The Application of the Calculus of Probalities to Statistics" in the *Bulletin of the International Institute of Statistics* for 1909.

[2] Suppose a market, with ten Xs on one side, ten Ys on the other; if, as above supposed, their natures or dispositions were constant from one (market-) day to another, the price determined by the play of competition would presumably be much the same from day to day, the range of indeterminateness being very small. Suppose next that the demand-and-supply schedule for each individual oscillates from time to time; the steadiness of the rate of exchange (though not its determinateness) would be impaired. But suppose further that the numbers on each side are raised from ten to a hundred. Then steadiness may be restored by the principle of averages. But alone it is not sufficient to secure steadiness, since it does not secure determinateness; for instance, in the case of two combinations, their average motives might be constant, but their bargains might vary from time to time.

arrangement—by an unconscious pressure of general convenience of the kind usually supposed to have brought about the adoption of money. No doubt this cause will explain something, but not, I think, everything. It will explain why a monopolist, apart from law and ethics, would probably not want to make separate terms with every individual, as strict theory might suggest. But it will not explain why in a monopolistic regime there may be several prices for (practically) the same article, but only one in a perfect market. To have analysed the conditions of a perfect market will be of assistance when we come to deal with monopolies and imperfect markets.

Composite Exchange.—A separate heading is scarcely required for the generalisation of the preceding theory. By supposing that the Xs offering x deal not only with Ys offering y, but also with Zs offering z, Ws offering w, and that dealers in each of these groups deal with two or more of the remaining groups. As Mill says, after analysing a certain case of simple exchange: "trade among any number of countries and in any number of commodities must take place on the same essential principles as trade between two countries and in two commodities." [1]

Economic Mobility.—The groups of Xs, Ys, etc., which supply different articles, may be imagined occupying different islands, between which there is a great gulf fixed, so that they which would go from one to the other island cannot. It is not possible for an X to assume the part of a Y, or vice versa. There is on this supposition no equation between the labour required in order to supply a certain quantity of x and the labour required to supply the quantity of y which is exchanged for that quantity of x. The quantities of labour, it should be remarked, must in the present section in strictness be conceived as measurable by an objective standard—so many foot-pounds, for example. But it is not easy, nor I think, usual, to treat Economics proper with such abstract rigour, as entirely to keep out of sight the cognate proposition—strictly belonging to our Section II *—that in the absence of mobility there will be no comparison between the trouble and benefit (in a subjective sense) resulting on an average to the parties on the different sides of the channel. On one side may be fruits growing wild, or quails rained from heaven, obtainable with a minimum of labour—with (practically) no labour at all if we suppose the amount of the commodity obtainable (per diem) to be limited. The denizens of this favoured

[1] *Political Economy*, Book III. chap. xviii. § 3; referring to International Trade.

* See added note on p. 368, above.

island may be luxuriating in superfluity, while their customers have to toil hard in order to obtain the necessities of a miserable existence.

Though exchange without mobility may appear the simpler conception, it is not that which the classical writers have primarily entertained. To illustrate their conception of value dependent on and proportionate to quantity of labour, we may conceive regions which are at times islands, at times connected by dry land. The market is held, say at the time of high tide, and goes on much as we have already supposed. But it is no longer possible that any X should permanently fare worse than if he were to be transformed into a Y; since it is now open to him to transfer himself, at low tide, from one island to another. The passage between two islands may be more or less free; sometimes, perhaps, only available at *neap* tides, so that a long period is required for a change in " market value " to lead to a change in " normal " value.

International Trade.—Exchange being the genus, and the absence of mobility the differentia of international trade, there is deducible from these essential attributes a characteristic property which may be enunciated as follows. The price [1] of the commodities imported in (relation to) the commodities exported is less than what it would be if, *ceteris paribus*, the trade were closed. For suppose that a nation of Xs being dealers of the sort above defined,[2] export x to and import y from a nation of Ys; if the price of y in x would be unaffected by the cessation (and reopening) of the foreign trade, there would be no motive for the continuance of the trade. This characteristic property includes as a particular case the classical principle of Comparative Cost. The property is more general as it applies to cases of value depending only on rarity, not (also) on effort and sacrifice. For example, before trade was opened between Japan and the Occident in the early 'fifties, the rate of exchange between gold and silver in Japan was about 1 : 6; while in the outer world it was about 1 : 15½. There was accordingly fulfilled the condition that gold should be exported from Japan to the Occident, silver exported from the Occident into Japan. The condition would have existed on the supposition that the value of the precious metal had nothing

[1] I follow M. Walras in thus employing price in the more general sense of rate-of-exchange.

[2] Otherwise if there were one or two predominant it might be in their interest and in their power to open the trade for the sake of obtaining the advantage of increasing returns by turning on to *one* commodity the productive forces which had hitherto been employed on *two*.

to do with cost of production, but depended entirely on rarity, or the total amount in existence (as possibly in the later Roman Empire). The re-statement proposed is not only more general, but less equivocal than the doctrine of Comparative Cost as stated by the earlier writers. For the Comparative Costs must be interpreted in the words of Professor Bastable [1] as " those which would exist at the margin on the hypothesis that each country is isolated; in the general case (not explicitly considered by the earlier writers) of cost varying according to quantity produced."

Distribution.—So far we have ignored the heterogeneity of the factors which co-operate in production. Our Xs and Ys might have been small independent artisans or peasant proprietors. We have now to take into account the stratification of modern industry—that the class of workmen is, as it were, on a different level from the class of landlords, and from the various strata which compose the capitalist employing class. We have to conceive, in the metaphor of Jevons, a horizontal as well as a vertical cleavage. If the vertical cleavage before represented by the isolation of islands is now represented by the separation between the wings and towers of a mansion, the horizontal cleavage may be represented by the separation between the flats which form the different stories of the mansion. In domestic or internal trade there is assumed to be more or less perfect mobility between the compartments on the same story; but not mobility between the stories, one of which may be imagined to contain the working classes, the other the capitalist employing class.[2]

In economics it is often difficult to hold fast general resemblances without ignoring—or appearing to ignore—specific differences. In the present matter, while apprehending that the transactions between the operative and the employing classes are the genus international trade, we must not forget that the exports and imports in this trade are of a very peculiar character. The peculiarity might be partially illustrated by the trade which

[1] ECONOMIC JOURNAL, Vol. XI. p. 228, last par., and p. 229, first par.; referring to the present writer's analysis at Vol. X. p. 391. *Cp.* Vol. XI. p. 589.

[2] The propriety of conceiving the transactions between employers and employed as a sort of international trade was discerned by J. S. Mill (*Political Economy*, Book III. chap. ii. last par.). Professor Pigou seems to sanction the view here presented when referring to the presentation of it in the ECONOMIC JOURNAL; he says: " Is it not the better view that the great divisions of the industrial world, land, capital, brain-power, trained hand labour, muscular labour, are now competing in the sense that against those who would pass from one to another there is a great gulf fixed " (*Edinburgh Review*, Vol. CCIII. p. 23, January 1906—an article of which Professor Pigou has acknowledged himself the author in his book on *Import Duties*).

used to flourish between England and the Southern States of America; these States exported to England raw cotton, receiving in return cotton manufactures. If the offer of raw cotton with the demand for cotton manufactures were to be increased on one side of the international market by a change such as the growth of population in the Southern States, other things being the same, the offer of manufactures on the other side of the market on the part of a large and flourishing England would be likely to keep pace with the offer of raw material, in such wise as not to alter the terms of international exchange to the disadvantage of the average Southerner. But, indeed, the illustration hardly does justice to the expansiveness of the trade which we are now considering. Let us rather suppose the export to consist of that rawest and most extensively demanded material, mechanical power. Let us imagine, for the sake of illustration, Niagara harnessed in the service of man to belong wholly to the United States, not in part to Canada; and that by improved means of transmitting force the means of production may be conveyed from Niagara to any department of Canadian industry. If the supply of power from Niagara to Canada were to be increased by some dislocation, for instance, some impediment to its supply elsewhere, then it might be expected that—in the long run, and abstracting temporary disturbance—the offer on the part of Americans owning Niagara would be met by the demand for additional power on the part of the entrepreneurs in a large and flourishing Canada.

This peculiarity of the quasi-international trade must be borne in mind when it is argued that the transition from a regime of Protection to Free Trade may place a portion of the working-classes in the position of a "nation" for whose exports there has ceased to be a demand. In thus arguing in the course of an exchange of views with Professor Bastable in the ECONOMIC JOURNAL, I ought to have emphasised the peculiarity which has just been illustrated. It is true that I guarded against misapprehension by comparing the increase in freedom of trade to an improvement in machinery, and quoting Mill to the effect that an improvement in machinery may be "very injurious to the labourers" on a supposition "purely ideal."[1] But I may have quoted, without sufficient reservation, Ricardo's conclusion that (in a certain supposed case) "population will become redundant and the situation of the labouring classes will be that of distress

[1] ECONOMIC JOURNAL, Vol. XI. p. 583. Cp. *ibid.*, Vol. X. p. 392, where the issue is described as "so minute a point in so hypothetical a case."

and poverty." Ricardo has " been supposing " (as he explains in the context) [1] " that improved machinery is *suddenly* discovered." Moreover, the probability of such a case was really different for Ricardo from what it is for us after a century's experience of improvements in machinery. Similarly, the proposition that the improvement in the mechanism of importation which is effected by the liberation of trade, though it may divert the direction, will (in the long run) not impair but increase the efficiency of the productive forces in the country, rests partly, I think, on specific experience of what improvements in arts of production have done for the benefit of the working classes. To distinguish empirical evidence of this kind from the still more diffused and universal experience on which the first principles of exchange (communicated in preceding paragraphs) are founded is, I think, philosophically just; and may be practically important in certain peculiar pathological cases.

Function of the Entrepreneur.—The central figure in the productive system is the entrepreneur. Buying the factors of production, the use of land, labour, machinery, and working them up into half-manufactured or finished products, which he sells to other entrepreneurs or consumers, at a price covering his expenses and remunerating his work and waiting. The symmetry of the entrepreneur with respect to the factors of production was first, I think, clearly enunciated by M. Leon Walras. Justly, or with very slight exaggeration, on the occasion of M. Walras's anniversary in June 1909, the leading economists and statisticians of Italy have declared: " The model that he was first to furnish for the comprehension of economic phenomena, the theory of economic equilibrium, constitutes the greatest advance which our science has received, since the impulse (*l'avviamento*) given to it by Ricardo." [2] All the more deserving of examination must be any tenet in the doctrine of such a teacher that challenges attention as paradoxical. In this spirit I have before now noticed the peculiar proposition that the entrepreneur normally makes neither gain nor loss. At first sight the dictum might pass as a *façon de parler*, as Companies make their liabilities and assets exactly balance by including among the liabilities of the Company the property of its members. But the latest utterance of the Lausanne school makes it indisputably clear that the proposition is to be interpreted literally. Professor Pareto recognises as fully as one could desire that the entrepreneur

[1] Ricardo, MacCulloch's edition, pp. 237 and 241.
[2] *Giornale degli Economisti*, June 1909.

is constantly pursuing his maximum advantage. But, we read, " cette fin elle-même peut se modifier par l'effet des moyens dont on veut se servir pour l'attendre." [1] The " curve of pursuit " along which the entrepreneur thus moves lands him in a position of null remuneration, through the action of competitors in the pursuit. " De cette façon les entreprises concurrentes, aboutissent là ou elles ne se proposaient nullement d'aller. Chacune d'elles ne recherchait que son propre avantage et ne souciait des consommateurs que dans la mesure où elle pouvait les exploiter, et au contraire par suite de toutes les adaptations et réadaptations successives enforcées par la concurrence toute cette activité des entreprises tourne au profit des consommateurs." [2]

Admitting that a process of the sort described may occur in certain phases of competition to which we are coming,[3] I do not recognise the description as typical of the entrepreneur's function in general, in a condition of stable equilibrium, or steady flow. In the pursuit of his maximum advantage he is rather to be compared to the lover on the Grecian vase, celebrated by Keats, who, though winning near the goal, does not advance—but does not therefore recede. Professor Pareto puts the case of an entrepreneur renting land from a landlord and producing wheat.[4] " If the entrepreneur has a monopoly he will procure for himself the maximum of advantage or profit (*bénéfice*)." [5] Agreed; assuming, what is indeed not explicably affirmed, that there is a plurality of competing landlords. " If there is competition between the entrepreneurs," the proprietor " will take all the *bénéfice* of the production and the entrepreneur nothing." Again agreed; assuming, what is now affirmed, that there is a single landlord and a plurality of competing entrepreneurs. But neither of these assumptions is appropriate to the normal case of Production and Distribution. We ought to assume a plurality both of landlords and entrepreneurs, like the Xs and Ys in the above paragraphs. In this typical case I cannot see that " lorsqu'il y a concurrence entre les entreprises celles-ci doivent se tenir sur les transformations complètes; elles n'ont donc ni profit ni perte." There is probably more than I have been able to apprehend in Professor Pareto's doctrine concerning the " complete transformation " of the factors of production by the entrepreneur [6] in a regime of competition. But I cannot see how it is inconsistent with the theory that the wheat produced flows partly into the hands of the entrepreneur, partly into the

[1] *Manuel d'Économie Politique*, ch. v. § 11.
[2] *Loc. cit.*, § 74.
[3] Subdivision B.
[4] *Loc. cit.*, § 62.
[5] § 64.
[6] *Loc. cit.*, chap. iii. §§ 75, 150 *et passim.*

barns of the landlord, in a proportion determined by the play of demand and supply; with no presumption that, in general, the proportion is extremely unequal, and tends to be unity to zero (or the reverse).

The conception of a set of cultivator entrepreneurs dealing with a set of landlords has been employed by me in criticising another paradoxical doctrine about the function of the entrepreneur, propounded by other eminent writers, on another continent. These writers do not regard the remuneration of the entrepreneur as null, but as precisely equivalent to the loss which would be occasioned to the community by the subtraction of an entrepreneur. The proposed equation would, as it happens, be verified by the particular case above instanced; but not, I have maintained, in the general case when instead of, or in addition to, landlords we have the owners of other factors of production, in particular hired labour.[1]

It will be understood that it is only the exact mathematical analogy between the marginal productivity of the entrepreneur and that of an operative in a large establishment which I dispute; I do not deny, I all along imply, that there is a general resemblance between the motives and winnings of the employer and employed, sufficient to justify Dr. Marshall's doctrine, that there is a supply price for business power.[2]

Nor do I dispute that part of Professor Pareto's exposition, which is in accordance with Dr. Marshall's doctrine, that the net advantages of a business are balanced against the incident efforts and sacrifices, in prospect rather than in the result at the stages of "quasi-rent."[3] Professor Pareto may be quite right in his surmise[4] that the actual remuneration of employers on an average is very small. It is the deduction of that tendency to zero from the theory of Competition that I cannot follow. Let me take another illustration. Suppose a set of competing landlords as before, each owning a portion of the land, which may be supposed limited in quantity and all of the same quality. But instead of competing cultivators, suppose now citizens competing for sites of bungalows and gardens in which to take holiday. We have once more a typical case of simple exchange between

[1] Above, p. 339 and context.

[2] *Principles of Economics* under the head of "Business Management."

[3] *Ibid.*, Book VI. chap. viii. (4th edition).

[4] *Manuel*, chap. v. t. 69, referring to certain Belgian Statistics. See the Hungarian statistics compiled by Korösi, to which reference is made in the ECONOMIC JOURNAL, Vol. XII. p. 251, and compare Marshall, *loc. cit.* Book VI. chap. viii. § 8, which may be considered the *locus classicus* on the subject.

Xs and Ys, a marginal price affording a maximum of advantage in the sense above explained.[1] That character of a market would be maintained even though it were true, in accordance with the shrewd remark of Sidgwick, that human beings have a tendency to over-value leisure as a source of happiness.[2] I am aware that Professor Pareto would not admit the parallel suggested between Consumers' and Producers' surplus.[3] But with reference to the purpose in hand I fail to see the difference. However, I am quite prepared to find that there is no material disagreement between us, that we are looking at different sides of the same shield—I at the gold side, he at a side which is devoid of all precious metal.

Marginal Productivity.—I go on to consider some objections which, though not so serious, on a question of pure theory, as the dissent of a great mathematical economist, yet deserve and may reward attention. I refer to the difficulty about the function of *Margins* in Economics which is felt by some who have not specially attended to maximum problems. No one has expressed these difficulties more strikingly than Mr. J. A. Hobson. The following extract is typical of his objections to Dr. Marshall's doctrine of the " Marginal Shepherd."

In order to measure the productivity of the last dose of labour let us remove it. The diminution of the total product may be 8 per cent. This 8 per cent. according to Marshall's method we ascribe to the last dose of labour. If now, restoring this dose of labour, we withdrew the last dose of capital, the reduction of product might be 10 per cent. This 10 per cent. is regarded as the product of the last dose of capital. Similarly the withdrawal of the last dose of land might seem to reduce the product by 10 per cent. What would be the effect of a simultaneous withdrawal of the last dose of each factor? According to Marshall's method clearly 28 per cent. But is this correct? " [4]

Quite correct, I have elsewhere replied,[5] adding, with reference to objections continued in the same vein as the passage above quoted: Imagine " an analogous application of the differential calculus in physics. . . . An objector substituting x wherever a mathematician had used dx or Δx."

To which Mr. Hobson retorts: " Professor Edgeworth appears to think that the Differential Calculus will assist him to find the

[1] Above, p. 368 *et seq.* [2] *Political Economy*, Book III. chap. vii.

[3] According to Professor Pareto the producer is obliged to remain on the " line of indifference " corresponding to " complete Transformation " and null remuneration. " Il y a là une différence essentielle avec les phénomènes qui se réfèrent avec gouts "—*Manuel d'Économie Politique*, chap. iii. § 79, and context).

[4] Hobson, *The Economics of Distribution*, p. 146.

[5] *Quarterly Journal of Economics*, Febuary 1904, p. 167.

productivity of the Marginal Shepherd by starting from the productivity of an infinitesimal margin of him." [1]

To which I rejoin that the introduction of Δx in conjunction with dx in the passage above quoted was not without significance. There was understood the presumption which must be borne in mind wherever, as in the present papers, we deal with applications of the Differential Calculus to Economics—that propositions true of *differentials* may often be extended to *small finite differences* [2] with sufficient accuracy for practical purposes. Such is the proposition now in question, that the effect of taking (or rejecting) in succession two or more small doses of different factors of production is sensibly the same as taking (or rejecting) both the doses together. Mr. Hobson has presented a contrary result by selecting for his illustration too large doses, not typical of the fine adjustments practised by the organisers of modern industry. A similar objection may be taken to an illustration which he has given in his latest treatment of the subject,[3] where the tenth dose, the difference between a "ten group" and a "nine group" of shepherds adds twenty sheep, while the eleventh dose adds only five sheep. There is a discontinuity between the returns to successive doses which is hardly typical of an establishment in which many operatives are employed.

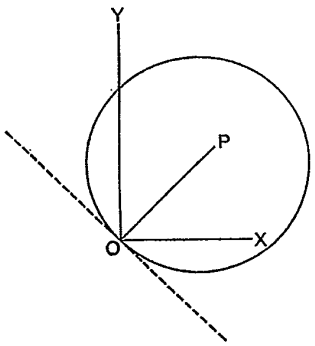

FIG. 2.

Let me take my own illustration. Let there be two factors of production x and y (*e.g.*, common labour and a certain kind of machinery); and let the amounts of each factor which a certain entrepreneur employs be respectively x and y. Let these amounts be measured from the point O along the abscissa and the ordinate respectively of Fig 2. Let the net return—the difference between the gross receipts and the total outlay [4]—be represented by z,

[1] *Journal of Political Economy*, 1904. Vol. XII. p. 460.

[2] In virtue of that continuity of functions which is commonly experienced. Compare ECONOMIC JOURNAL, Vol. XVIII. 1908, p. 399; and below, p. 389.

[3] *The Industrial System*, p. 110.

[4] We might represent the *gross* receipts by adding to z as above defined $kx + ky$, the perpendicular distance from the point (y, x) to the corresponding point in a plane above the paper, whose section with the plane of the paper is

the perpendicular distance measured downwards from any point (x, y) in the plane of the paper to the corresponding point on the (concave) surface of a spherical—or rather hemispherical—bowl underneath, the rim of which is the circle shown in the figure (being a " great circle " of the sphere of which the bowl forms a hemisphere). Thus initially when x and y are each zero, z is zero. As x and y are increased z will increase up to the maximum which occurs at the point P; for which if R is the radius of the sphere and of the (great) circle shown in the figure, $x = \frac{1}{\sqrt{2}}R$, $y = \frac{1}{\sqrt{2}}R$. Now let us consider the effect of removing—first successively, then simultaneously—small doses of each factor, say one per cent. of each, or, as more convenient to calculate, while less favourable to our thesis, $\sqrt{2}$ per cent. of each that is 0·01R. Whereas R measured the maximum net profit, the diminution of profit caused by reducing the abscissa from $\frac{1}{\sqrt{2}}R$ to $\frac{1}{\sqrt{2}}R - 0{\cdot}01R$, the ordinate remaining unchanged, is found to be

$$R - \sqrt{R^2 - 0001R^2} = R(1 - \sqrt{0{\cdot}9999}).$$

Likewise the diminution of the ordinate by $\sqrt{2}$ per cent. thereof is R $(1 - \sqrt{0{\cdot}9999})$. The sum of these two effects, viz. 2R $(1 - \sqrt{0{\cdot}9999})$ is to be compared with the effect of taking the two doses together, viz. R $(1 - \sqrt{0{\cdot}9998})$. It appears that while the former subtrahend is 0·0001000014..R, the latter is 0·000100005..R—not a very important difference, per cent. of net profits (or gross receipts). The difference will still be insignificant even when we take away doses so large as 0·1R, that is, above fourteen per cent. of each factor. The sum of the effects of (removing) the two doses separately is now 0·010025..R; the effect of the two doses together is 0·01005..R.

Of course if we go on increasing the size of the doses, we shall reach a case in which the difference under consideration is significant. Thus if all industry were organised on a scale in which no entrepreneur could take on more than one or two

---

the tangent to the circle at O, its equation being $z = kx + ky$. Of course in general the coefficients of x and y would not be equal, the line in which the plane thus designated cuts the plane of the paper would not make a half right angle with each of the axes. The coefficient k may be interpreted as the price of the corresponding factor, " accumulated," as explained by Dr. Marshall, so as to be comparable with prospective receipts.

employees, the remuneration of the last employee taken on would very imperfectly measure—might afford only a very superior limit to—the remuneration of an employee. In fact that remuneration would be indeterminate (in the mathematical sense) within certain limits;[1] unless indeed under the appearance of selling work by the day there really lurked a traffic in hours of work,[2] and so the requisite comminution of doses reappeared. Except, of course, also the cases in which the remuneration of the hired operatives is equated by competition to the remuneration of small independent artisans, of settlers on free land, and so forth. In short, I do not deny that there *might* be an industrial system such as Mr. Hobson seems to suppose in which the marginal productivity of an operative does not (with any accuracy) measure wages. But I opine that such a system is not typical of modern industry in the concrete. And I affirm, what Mr. Hobson in the passage above quoted seems to deny, that the following theorem is true in the abstract. If the labour market consisted only of a number of employers on one side, and a much greater number of employees all of the same efficiency on the other side, with no limitations on the size of establishments, wages would be approximately measured by the marginal productivity of an operative.[3] This simple corollary to the doctrine of maximum advantage does not seem to be touched by Mr. Hobson's dialectic, of which I subjoin some more specimens.[4]

"Margins are derivative, not determinative."

"This [the author's doctrine] does not imply that the marginal factors exercise any special determinant influence as causes of the price."

"It is far more accurate to say that the price per unit causes the margin to be where it is than to attribute any causative power to the margin, as margin, in relation to the price per unit."

Does this dialectic add anything to the warning which Dr. Marshall in connection with his doctrine of the Marginal Shepherd had clearly pronounced? "It must, however, be remembered that the price which at first was worth while for the farmer to pay for this labour merely gauged the outcome of multitudinous causes which between them govern the wages of shepherds; as the movements of a safety-valve may gauge the outcome of the multitudinous causes that govern the pressure in a boiler."[5]

[1] Index, s.v. *Determinate Equilibrium.*

[2] *Loc. cit.*, p. 189.

[3] The law of diminishing returns, proper to the case here designated A, being postulated.

[4] *The Industrial System*, p. 102.

[5] *Principles*, Book VI. chap. i. § S. (3rd ed., 1895), and context.

I submit that difficulty and logomachy might be avoided by contemplation of a physical analogue, the principle of maximum energy. Suppose in the construction pertaining to Fig. 2 that a weight sliding down towards the lowest point of the bowl from the rim does so many foot-pounds work. The point at which the maximum energy will have been put forth, the position of the minimum potential energy, or equilibrium, is (corresponds to) P. Is it necessary to inquire whether the last step of the path which stops at P is " derivative " or " determinative " ?

Incipient Taxes.—An interesting example of the transition from differential to small finite differences—where what is strictly true *de minimis* is predicated also *de parvis*—occurs in the remarkable paper of which I have here borrowed the title.[1] The author, Mr. C. F. Bickerdike, argues that theoretically in normal circumstances it is always possible for a country to obtain a net advantage by the imposition of small import or export duties without supposing (as usual when such admissions are made by economists) that the country has a kind of monopoly as seller or buyer. The argument is based on the proposition that the loss of *consumers' surplus* occasioned by the disturbance of industry is of a negligible order as compared with the gain to the Treasury.

This proposition is a particular case of a wide principle which may thus be worded : " A small change of an economic variable quantity at the margin commonly causes a *very small* change in the corresponding surplus." The principle is not confined to Economics, being based on the general theory of maxima. Good examples are afforded by astronomical quantities of which the variation is recorded in the Nautical Almanack. For instance, the distance of the sun at noon (along a great circle) from the celestial equator varies more rapidly when the distance is near its maximum, about the times of solstice, than at other times. Thus the variation in the sun's declination[2] per hour at the time of the summer solstice (at noon of the day on which the solstice occurs) is 0·36 second. The variation at the time of the spring equinox per hour is 59·25 seconds. The change is more than 160 times greater at the equinox than what it is at the solstice. If we consider a degree (or small finite difference) of time larger than an hour, say a day, there will still exist the same kind of

[1] " Incipient Taxes," by C. F. Bickerdike, ECONOMIC JOURNAL, Vol. XVI. p. 529; *cp.* Vol. XVII. p. 98.

[2] The *apparent* declination; Nautical Almanack, 1909. The variation per hour, at noon of the day on which the hourly variation is a minimum, is larger or smaller according as the actual moment of solstice is further from or nearer to noon.

contrast, but less in quantity—as theory leads to expect. The change in declination per day at the spring equinox is very much greater than what it is at the summer solstice.* Like propositions are true of the winter solstice, and all similar cases. The principle may be extended from variations of *quantities* to variations of *form* (the special object of the Calculus of Variations).

This principle seems to be of wide application in Political Economy, more especially in the larger sense of the term to which we are coming in our second section.** It has been employed as an argument in favour of the diffusion of taxation over a great number of commodities.[1] It may be employed to justify small interferences with the natural course of industry for the sake of large ulterior advantages. A small readjustment of the entrepreneur's plan of production and "margins of profitableness," recommended for some considerable non-economic advantage, becomes additionally advisable by the theorem that the economic loss is likely to be not merely small, but *very* small.[2] No doubt it is a dangerous doctrine to enunciate; very liable to be abused by Protectionists and other unscrupulous controversialists, who if you give them a *differential* will take an *integral*.

* See Index, s.v. *Maximum*.

** See introduction to this article above, p. 367.

[1] Above (δ), p. 351, referring to δ. The argument is no doubt overborne by practical considerations on the other side.

[2] See on the principle of incipient taxes. Above, δ.