(Y)

GRADUATION OF TAXES

[REPRINTED from the ECONOMIC JOURNAL, 1919, where the fuller title describes the taxes which are to be graduated as taxes on income and capital. It is argued that the simple scheme proposed by Cassel is not appropriate to the very high taxation now prevalent. For the calculation of the tax from the taxable amount Multiplication and Division must now be supplemented by Involution or Logarithm. Not otherwise can there be realised the two conditions, the first productional and the second distributional, (i) that the taxpayer should not be deprived of motive to increase his income, (ii) that the rate of taxation should continually increase with the increase of the income. The use and purpose of "graduation" are more fully described in the following paper (Z) and the introduction thereto.]

METHOD OF GRADUATING TAXES ON INCOME AND CAPITAL

Among the formulæ known to me as having been suggested for the purpose of graduating taxation, a foremost place is due to the scheme proposed by Professor Cassel in the ECONOMIC JOURNAL.[1] Varying his notation, we may write

(i) $$T = r(X - E);$$

where T is the amount of the tax (in pounds sterling, or other monetary unit); r is a percentage or (decimal) fraction; $X - E$ is the taxable income; E is an abatement, not a fixed minimum, as Mill proposed, but varying with the income—not in an opposite sense as in many contemporary systems, but *increasing* with the increase of income.

(ii) $$E = \frac{XM}{X + M - e};$$

where e is the *minimum* of subsistence below which the tax does not descend, *e.g.*, £130 in the present British income-tax; M is the maximum abatement, a limit which is more and more nearly

[1] Vol. XI. (1901), p. 485 *et seq.*

approached (but never reached) as X increases. Substituting the value of E in the expression for T, we have

$$\text{(iii)} \qquad T = r\frac{X(X-e)}{X+M-e};$$

an expression which becomes zero, as it ought, when $X = e$.

Distinction may be claimed for this scheme on the following grounds :—

(1) It is elementary, " intelligible to the most untaught capacity," a great merit in a principle of currency according to Mill, and doubtless some merit in a principle of graduation.[1]

(2) It exhibits a mathematical elegance, which is also a fiscal excellence,[2] in that it is capable of representing a great variety of tax systems by means of a very few adjustable coefficients or " constants."

(3) Of its constants two, e and M, are determinable *a priori*, so to speak, from a knowledge of the people's wants and habits; the third, r, being adjustable according to the needs of the Treasury.

The *first* merit is conspicuous. The formula involves only the common arithmetical processes; the operation which is highest in a mathematical sense being *division*.

To illustrate the *second* feature I proceed to show how the formula is adaptable to actual tax systems. The first scale which I adduce is one relating to the continent of Europe before the war. The scale is constructed from the statistics of income-taxes in several European States as presented in a Blue-book dated 1913.[3]

TABLE I.—PRE-WAR CONTINENTAL INCOME TAX.

Income	£40–	80–	100–	150–	300–	500–	1,000
Tax per cent.		*2·64*	*2·92*	*3·47*	*4·29*	*4·84*	*5·11*
Income	£1,500–	2,000–	3,000–	5,000–	10,000	Maximum.	
Tax per cent.	*5·53*	*5·84*	*6·29*	*6·66*	*6·98*	*7·2*	

Each rate in this table is obtained by taking a Mean—that mean which is called the Median—of the rates pertaining to an assigned amount of income in each of several States. For this purpose several Swiss Cantons have been lumped together so

[1] Mill, *Political Economy*, Book III. chap. xiii. § 2. The condition is less imperative in the case of taxation, inasmuch as the mathematical basis on which the contribution of the taxpayer is calculated need not be obtruded on his notice; it suffices that the authorities should promulgate an *arithmetical* schedule of the amounts payable on each amount of income or capital.

[2] As pointed out in *A Levy on Capital* (by the present writer, 1918), p. 85.

[3] [Cd. 7100.]

as to count as one State. Also three minor German States have been similarly treated. For example, in order to determine the figure which is to be put for the rate of taxation of an income of £100 (up to £150) I utilise the following data :—

Prussia.	Bavaria.	Other German States.	Denmark.	Norway.	Sweden.	Holland.	Switzerland.
3·19	2·65	3·20	3·80	2·39	2·27	2·27	4·82

The "Other German States" are Saxony, Wurtemberg, and Baden, with rates respectively 3·00, 3·20, and 4·25; whereof the *second* in the order of magnitude is taken as the Mean. Likewise, 4·82 is the Median (half-way between the third and fourth in the order of magnitude) of the rates for six Swiss Cantons. The Median of the eight figures thus obtained is 2·92 (half-way between 2·65 and 3·20). The exempted minimum for the majority of the States appears to be 40; and accordingly, I take that for the value of e. But as the tax for some States does not descend to 40, I have not formed a mean value for the rate between 40 and 80. At the other extremity the fixed proportion designated by r in the formula is evidently 7 per cent. (approximately).

As to M, I have not attempted to verify the *third* claim by determining this constant, as theoretically possible, from the conditions of Continental life. For the purpose of illustrating the adaptability of the formula, it suffices to determine M from the condition that for some assigned income the rate given by the formula (with the two given constants) should be the actual rate shown in the table. Consider, for instance, the income £1,000, the rate against which in the table is 5·11 per cent., the tax therefore being £51·1. We have then by equation (iii) :

$$51{\cdot}1 = 0{\cdot}07\frac{1000 \times 960}{M + 960},$$

whence $M = 355$. If we had taken the rate for 2,000 as the datum, we should have the equation :

$$2000 \times 0{\cdot}0584 = 0{\cdot}07\frac{2000 \times 1960}{M + 1960}$$

whence $M = 390$.

If we put for M the nearest round figure, 400, that will be found, with the other constants, to give fairly good results. For instance, for the income £1,500 the tax as calculated by the formula is £82·4; actually it is £82·9. For income £5,000 the tax calculated is £324; actually it is £333.

The formula fits well many other pre-war tax systems, characterised by the feature that as the income increases indefinitely, the rate approaches a fixed and small proportion.[1]

But when we turn to war income-taxes we find that the ultimate fixed proportion is no longer a small percentage. Thus the British Income-Tax as modified by the Budget of 1918, rises to above 50 per cent. From the new scale as given in full by Mr. W. M. J. Williams in the *Journal des Économistes* [2] I select some specimen data. The earlier figures relate to "wholly unearned income." For the later figures income-tax (at 6*s.* in the £) and sur-tax are combined.

TABLE II.—BRITISH INCOME-TAX, 1918.

Income in pounds.	Tax in pounds.	Shillings per pound.
200	12	1·2
1,000	187·5	3·9
10,000	4,187·5	8·4
20,000	9,437·5	9·5
40,000	19,937·5	10·0
100,000	51,437·5	10·3

Proceeding as before, let us put 50 per cent. as (approximately) the ultimate fixed proportion, while for e we have 130. From these data there follow inferences as to the abatement which are not consonant with the *third* of the merits above claimed for the Cassel formula. In accordance with equation (i) put $T = 0{\cdot}5(X - E)$. Then in order that the equation may be satisfied when $X = 10{,}000$, we have $4187{\cdot}5 = 0{\cdot}5(10{,}000 - E)$. Whence E, the abatement, is £1,625; rather a high figure for necessaries! But it is not the highest figure implied. Employing equation (iii) we have:

$$4187{\cdot}5 = 0{\cdot}5\frac{10{,}000 \times 9870}{9870 + M},$$

whence $M = 1915$! [3]

If we are to abandon the *rationale* of Professor Cassel's formula, and to treat it as simply empirical, a further simplification may be advised. Let us no longer treat the tax as a *function* of the abatement. On that arrangement if the taxpayer is, in

[1] Seven per cent. in the example above given, eight per cent. in the example worked by Prof. Cassel (with a somewhat different notation) in the ECONOMIC JOURNAL for 1901, p. 491.

[2] June, 1918, p. 316.

[3] The practice of the English Law with respect to the "necessaries" of "infants" may be referred to as justifying some extravagance in the estimate of what is necessary for persons in a high station of life. See Anson, *Contracts*, *sub voce* "Infants."

Latin idiom, increased by a child, and obtains a corresponding increase of exempted income, an entirely new schedule has to be calculated. There would be as many schedules as there are varieties of abatement. But it is much simpler to treat the tax as a function of the surplus of the taxable income over and above the deducted abatement. There is thus room for the greatest variety in the grounds for abatement: children, wife, insurance; perhaps invalidity, perhaps change in the value of money, perhaps station in life.[1]

This change is easily effected by putting $e = 0$ in the above written formula (iii), and for X (the total income) x, the surplus above the untaxed abatement, which does not now figure in the formula. The formula thus generalised may be written:

$$\text{(iv)} \qquad T = x\frac{rx}{M + x}.$$

For instance, utilising data furnished by the British income-tax for 1918, let us determine M and r from the equations:

$$\text{(1)} \qquad r\frac{39870^2}{M + 39870} = 20{,}000,$$

$$\text{(2)} \qquad r\frac{870^2}{M + 870} = 187{\cdot}5,$$

from which I find $M = 1217$, $r = 0{\cdot}517$. Applying the formula thus determined to an income of £100,000, that is, a surplus of taxable income of £99,870, I find for the tax £51,140—much the same as the actual tax, £51,437 10*s*.

Is there any reason to think that we should fare better with any other formula involving only *three* constants (two in addition to the abatement, which is not *explicit* in the formula as now modified)? We shall be better able to answer this question after considering two defects which may be attributed to the formula, whether in its original or its generalised shape.

First, the formula is not suited to represent very steep graduations; the case when the rates of taxation increase very much more rapidly than the taxable incomes. Let x_1 and x_2 be two taxable incomes, the latter being the greater; and let ρ_1 and ρ_2 be the corresponding rates of taxation. Then by hypothesis, since the tax is to be progressive, ρ_2 is greater than ρ_1; say, $\frac{\rho_2}{\rho_1} = q$, where q is an improper fraction. Substituting for ρ_1 and ρ_2 their

[1] The Australian Commonwealth appears to be particularly select and generous in the specification of grounds for exemption. See Commonwealth Report cited below.

values obtained from equation (iv) (and remembering that $\frac{T}{x} = \rho$), we have $\frac{\rho_2}{\rho_1} = \frac{M + x_1}{M + x_2}\frac{x_2}{x_1}$. Whence it follows that q is less than $\frac{x_2}{x_1}$. But this is a limitation upon the progression which may be undesirable. It may be required that, as in the present American income-tax, while the tax on £1,000 is £16, the tax on £2,000 should be £71 [1]; and accordingly that, while

$$\frac{x_2}{x_1} = 2, \frac{\rho_2}{\rho_1} = \frac{35{\cdot}5}{16} = 2{\cdot}2 \ldots$$

It may be pleaded that such steep graduation is abnormal. But it is doubtful whether any norm or standard can be prescribed for the income-tax as distinct from the *tax system* of a country. For the income-tax is usually complementary to other parts of the system, in particular to taxes on commodities and local taxation. Where the taxes on commodities were very heavy—pressing most heavily on the lower incomes—such a scale as that which has been instanced might well be appropriate. A formula adopted to general use ought to be better guarded against the objection which has been exhibited.

But grant that this objection is not very serious, especially with respect to taxes on capital. Admit that the formula under consideration affords as good a fit as any other function involving only three constants, to the taxes on income and capital which are in actual use. Yet adaptation to existing forms is not the sole test of the adaptability which we require. Our task is not exactly that of the statistician who employs a favourite formula to represent a concrete set of data—a given " histogram." Our part is not so much that of the portrait-painter as of one who draws ideal " subjects." Our formula should be adapted to represent graduation, not only as it is, but as it ought to be. Now the Swedish designer of fiscal forms falls short of ideal perfection at one point. He may be contrasted with the sort of artist that was to be found in Rome, capable of modelling hair and nails to perfection, but unsuccessful in the composition of a whole.[2] Contrariwise, Professor Cassel's work as a whole is admirable. But he fails to represent one extremity in its ideal perfection. He copies it indeed perfectly as it actually occurs, compressed and deformed like a Chinese lady's foot. Such, I submit, is the

[1] It is true that in the actual tax the £1,000 and the £2,000 include the abatement, and so correspond to our X, not our x, but it might have been otherwise.

[2] Horace, *Ars Poetica*, 32.

character of the graduations commonly in use which approach, but never pass, a certain finite rate. Can any good reason be given for thus exempting the higher incomes and capitals from progression? Surely the exemption has not been adopted by officials as a deduction from the principle of "equal sacrifice" in accordance with the ingenious reasoning of Mr. Cohen Stuart.[1] "As soon as all personal wants are pretty well satisfied," he argues, "the possession of income has no longer any influence on consumption. It is a figure the increase of which by a certain percentage would give about the same pleasure to a man with 10 millions of francs per annum as to one with 100 or 500 millions." Or is the reason one of those given by other theorists with less lucidity?[2] Could it be fear of alarming the millionaire, even when the final rate was so moderate as 7 per cent., as in the pre-war Continental taxes above cited? Was it a not unfounded belief that the condition of continual progression could not be secured by elementary arithmetical operations? Or simply poverty of mathematical resource?

II. Whatever may have been the reasons in the past for this lenity to millionaires, it may be doubted whether it will continue to appear reasonable in the future. There will be a demand for a formula fulfilling the condition of an effectual continual progression. The following formula seems to satisfy those conditions:

$$(\text{v}) \qquad x - T = ax^{\beta},$$

where, as before, x is the excess of income or capital above a specified minimum; T is the amount of the tax; $x - T$, say, y, may be described as the "available surplus," that which remains to the taxpayer (over and above the exempted minimum) after he has paid the tax; a and β are numerical constants, β being always fractional.

An example will form the simplest explanation of the scheme. The example is furnished by the American Federal Income-Tax of 1917.[3] I transcribe part of the schedule, commuting dollars into pounds sterling.

TABLE III.—AMERICAN FEDERAL INCOME-TAX.

Income......	£1,000	2,000	4,000	12,000	20,000	100,000
Tax.........	£16	71	236	1,356	13,236	38,536

From the information given I assume that £400 may be treated as an exempted minimum.

[1] Discussed by the present writer in the ECONOMIC JOURNAL, Vol. VII.: above, S, p. 110. [2] *Loc. cit.*, p. 109.

[3] As stated by the Guaranty Trust Company, New York.

To determine a and β we must utilise two of the data, say, the tax on £1,000 income and that on £12,000. We have thus two equations :

(1) $a(12{,}000 - 400)^{\beta} = (12{,}000 - 400) - 1{,}356 (= 10{,}244)$.
(2) $a(1{,}000 - 400)^{\beta} = (1{,}000 - 400) - 16 (= 584)$.

Whence (taking logarithms and eliminating a) I find for β, 0·967, and thence for a, 1·202, nearly. The construction will be found to fit fairly well at different points. For instance, for an income of £4,000 the calculated tax is £296, the actual tax is £236. For an income of £20,000 the calculated tax is about £3,000, the actual £3,236. Of course, if we had selected other points for an exact fit, we could have secured greater closeness of fit than now, in the neighbourhood of those points. But we cannot expect with only three constants at our disposal to obtain a good fit at all points.

There is one tract, however, for which it is not in general possible to secure a good fit, namely, the lower extremity. As the income diminishes, we come to a point at which the tax is zero; and if we descended below that point the tax would pass into a bounty! This limit is given by equating the available income to the total (untaxed) income above the minimum, *i.e.*,

(vi) $$ax^{\beta} = x,$$

whence $x = a^{1/1-\beta}$. Thus in the example just now given, if Log $a = 0{\cdot}079$, $\beta = 0{\cdot}967$, we have for the limiting value the number of which the logarithm is 2·4 nearly, *i.e.*, about 251. Which, added to 400, the minimum exempted, gives 651 for the figure below which the construction is not applicable.

III. The new formula seems specially suited to serve as a sur-tax. It may thus complement the Cassel formula when that fails at the upper extremity. At a certain point the new tax may be as it were yoked on to one of the Cassel type. To avoid a *jolt*, it should be arranged that at the point of junction the sur-tax should be zero.

To illustrate the composition of the formulæ I recur to the statistics of the American income-tax, and proceed to arrange that when the income has reached £2,000, a sur-tax of the kind described should be superimposed on a Cassel tax. For the calculation of the Cassel tax I make the convenient assumption that the highest abatement for "necessaries" which the American millionaire can claim, the M of the formula, is £800. As before, I take £400 as the minimum abatement. Then by equation (iii) for any assigned income, X, we have $T = r\dfrac{X(X - 400)}{800 + X - 400}$. This formula must give us the whole tax for an income of £2,000, since the sur-tax is to be zero at that point. Putting for X 2,000, and for T' the given taxation on an income of that size, viz. 71 (£), I find $r = 0{\cdot}05325$. Now let us take an income well above £2,000, *e.g.*, £20,000, and determine the co-efficients of a sur-tax so that it may both (1) start at £2,000, and (2) at £20,000 may prescribe a tax which, superadded to the Cassel tax for that income, may be equal to the given tax, viz., £3,236. First, for the Cassel tax with the constants above stated I find 1023·25. The sur-tax therefore should contribute (3236 — 1023·25), or 2212·75. That is, the available income (on the supposition that the sur-tax only acted) should be (19600 — 2212·75) or 17387·25. We have thus the two equations :

$$(1)\ \alpha 19600^{\beta} = 17387{\cdot}25.$$
$$(2)\ \alpha 1600^{\beta} = 1600.$$

Whence I find for β 0·952, and for α 1·425. It will be found that this construction gives a fairly good fit at points not too distant from those at which the fit is by construction exact. Thus for an income of £4,000 the tax is by calculation £311, actually £236.

Satisfactory as this result appears, the formula from which it is deduced cannot be accepted as universally appropriate. For it violates the canon that, however large the income or capital may be, the tax should not be such as to deprive the taxpayer of the motive to work and save. To be sure, in the instance given the breakdown is far enough off. The taxable amount would have to rise to some millions of trillions sterling before reaching the point at which an increase of the total income would result in a diminution of the available income. And very generally, if, as commonly, I think, it could be arranged that the fixed ratio r of the Cassel part of the formula should be small, not exceeding, say, 0·1 (10 per cent.), it may be expected that the breakdown is at a safe distance.[1] But possibly, and especially in a case above noticed,[2] the data may prove recalcitrant.

IV. To be safe from the danger which has just been indicated, it might be better to yoke the new formula with that of Professor Cassel, not abreast, so to speak, but *tandem*. Let the Cassel tax act by itself up to an assigned figure, say, as before, £2,000; and thereafter let the new tax by itself rule. We have only to arrange that the new formula should give the same figure for the tax on that income as the Cassel formula, namely, the given figure 71; and also that it should satisfy the datum for any other income,

[1] Compare note 3 to p. 252, below. [2] Above, p. 247.

say, as before, £20,000. We thus obtain two equations for the constants α and β, namely :

$$(1)\ \alpha(1600)^{\beta} = 1600 - 71.$$
$$(2)\ \alpha 19600^{\beta} = 19600 - 3236.$$

From which I find $\beta = 0{\cdot}946$; $\alpha = 1{\cdot}42$.

V. Another method of employing the new formula (introduced in Section II.) as a sur-tax is to take for the primary tax, not the Cassel formula, but one of the new type, that one which does not become a bounty.[1] This condition is secured by putting $a = 1$ in the expression for the available income; which thus becomes of the form x^b (b less than unity). At a suitable point there is to be either added to, or, better, perhaps, substituted for, this formula one of the more general type ax^{β}.[2]

The first arrangement is not perfectly safe. But the danger is not in practice, I think, to be much apprehended. Consider, for instance, the example given in the lecture above referred to (*Levy on Capital*). According to the formula there offered as representative of the present English income-tax, the "available" income, say y (*i.e.*, the amount in excess of the exempted minimum, say x, less by the tax), may be written :

$$y = x^{\cdot 974} + 1{\cdot}22x^{\cdot 962} - x.$$

The expression for y continues to increase with the increase of x, up to a value of x which is above £10,000,000,000 ![3]

A geometrical representation of these constructions is offered on p. 253.

The abscissa measured along the horizontal OA from the origin O denotes income or capital. The ordinate Xy corresponding to any abscissa OX denotes the amount that the taxpayer has at his disposal after paying the tax—including an exempted minimum. The ordinate can never rise above OB, a right line, making an angle of 45° with OA. OE denotes the exempted minimum; an abatement varying for different persons, according to the number of children, etc. The abscissa ox measured from o as origin denotes the taxable income. The ordinate xy denotes

[1] More exactly, does not become a bounty until the taxable income is less than £1.

[2] The expression for y the ordinate which represents the available income becomes on the first plan $x^b - T$, where T is the sur-tax; $= x^b - (x - ax\beta) = x^b + ax\beta - x$. Accordingly $\frac{dy}{dx} = bx^{b-1} + \beta ax\beta^{-1} - 1$; an expression which may ultimately become negative; just as the compound formula of Section III.

[3] Analogously the example given below in Section VII., though manufactured to exemplify difficulties, has no terrors for the present method, which would continue to be applicable up to incomes over £100,000,000 !

the available surplus, being the taxable amount minus the tax. The length intercepted between y and the line oB represents the tax. A right line, oC, dividing all ordinates in the same proportion, represents a uniformly proportional tax above a certain exempted minimum (Mill's ideal). The curve-line oy is intended to represent a tax according to the formula of Professor Cassel. It will be observed that the rate of taxation (whether relatively to the total or the taxable surplus) continually increases. The abatement, too, continually increases. For, by equation (1), the abatement $E = (rX - T) \div r = (rx - T) \div r + e$. Now rx is denoted by TR in the figure; and T by yT; and the curve is such that yR (as well as Ty) continually increases.

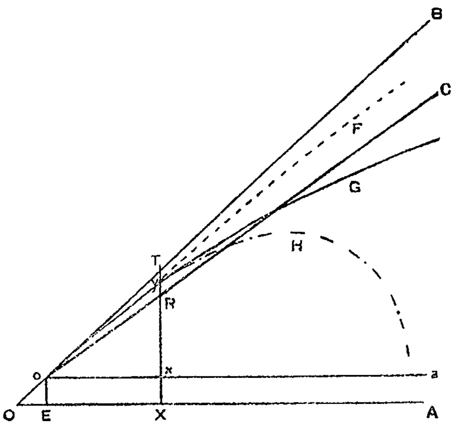

Fig. 16.

The Cassel tax is supposed to function independently up to the point y in the curve corresponding to x on the abscissa. At that point the new tax is substituted. Beyond that point the dotted curve yF represents the continuation of the Cassel curve, the rate continually approximating to r; the vertical distance of the curve from the right line OC approximating to the limit $r\ (M - e)$. The thick curve-line beyond y, yG, represents the new tax, as employed by itself in Section IV. The tax is such that y, the available surplus, continually increases; while, at the same time, $T \div x$, the rate of taxation, also increases up to the limit of 100 per cent. The *broken* curve, which also diverges from y, represents the compound tax constituted by superimposing the new formula upon that of Professor Cassel after the manner shown in Section III. The *hump* at H in the curve, representing the available surplus, is designed to illustrate the particular case in which the compound formula would be inappropriate.

The figure also serves to illustrate Section V. The thick curve to the right of the point y may still represent the new formula as substituted in Section IV. The line to the left of y is suited to represent the curve $y = x^b$ as well as the curve which designates the available surplus according to the (generalised) Cassel formula; for both of which curves $\frac{dy}{dx} > 1$, $\frac{d^2y}{dx^2} < 1$.

So far, we have taken no account of the circumstance that the number of persons enjoying an assigned income or capital varies with the amount assigned. To represent this varying number, there would be required another dimension, a third axis—say z—perpendicular to the plane of x and y, the plane of the paper. The curve in the plane xz, which represents the distribution of incomes, may be expected to fulfil a well-known law due to Pareto.

VI. The systems which have been proposed encounter a formidable rival in a formula suggested by Mr. Douglas White.[1] He takes the exempted minimum as the unit of income. Then if the income measured in that unit $= X$ (to use our own notation), he in effect [2] puts for the rate of taxation (on the whole income) r Log X, an expression which reduces to zero, as it ought, when $X = e$ (the exempted minimum). Considering that only two constants are here employed, r and e in our notation, the success which Mr. White has obtained is remarkable. But it is not greater, I think, than that which attends our new formula (introduced in Section II. above) when limited to two constants (*including* the abatement) by putting $a =$ unity, as in Section V. The formula thus presented has the advantage of not involving the exempted minimum. It is free also from defects which may be attributed to the White formula in common with a more general form to which we now proceed.

VII. Mr. White's formula may be generalised by employing a similar form, with a new constant, referring to the taxable income (above the exempted minimum); as thus,

$$\rho = r \operatorname{Log}\left(1 + \frac{x}{c}\right);$$

[1] ECONOMIC JOURNAL, Vol. XXI. (1911), p. 371 *et seq.*

[2] I must apologise to Mr. White and other authors for making rather free with their notations and conceptions for the purpose of the comparisons here instituted.

where x is now, as before, the taxable surplus and ρ is the rate of taxation on that surplus; c is a new constant. For example, to obtain a graduation on the lines of the American income-tax, let us operate on the data for incomes of £1000 and £50,000. We have then (putting for e, as before, 400) the two equations:

$$(1) \qquad r \operatorname{Log}\left(1 + \frac{49600}{c}\right) = \frac{13936}{49600}$$

$$(2) \qquad r \operatorname{Log}\left(1 + \frac{600}{c}\right) = \frac{16}{600}.$$

Easily eliminating r, we obtain an equation for c which is approximately satisfied by $c = 1,600$. The corresponding value of r is roughly 0·19. The formula thus obtained will be found to fit the given scale at different points fairly well.

But the construction will not work so well in all cases. It is open to the same objection as the Cassel formula that it is unsuited to represent very steep graduation. If $\frac{\rho_2}{\rho_1}$ is very large, larger than $\frac{x_2}{x_1}$, then it may not be possible to find a value of c which complies with the data. A more serious defect is the liability to excess of taxation at the upper extremity. To exhibit this, suppose it to be prescribed that the taxation of an income of £1000 should be what it is for the present British income-tax, namely, £187 10*s.* on £870 (the surplus above £130); but that for the smaller income of £200 the tax should be much less than what it is according to the British income-tax, say, instead of £12, only £2 or a trifle less (on £70). As above remarked,[1] we cannot be certain that a progression which looks anomalous may not be appropriate to a (complementary) income-tax. The constants which satisfy these conditions are (roughly) $r = 0·5$, $c = 512·4$. Accordingly, in the neighbourhood of the points utilised, the formula thus furnished complies with the conditions of a workable progressive tax. But consider a point at some distance from these tracts, above £50,000. For this size of income the formula gives a tax greater than the income! But the extent of the failure is not fully shown by this result. At a much earlier stage, namely, just above £6,000, the formula ceases to be admissible because by increasing his income beyond this limit the taxpayer would incur *loss*. The *broken* curve yH in the figure may serve to represent this failure. On the suppositions just now made, H would correspond to a taxable surplus just above £6000. The point corresponding to £51,000 would be *below* the axis oa!

[1] Above, p. 248.

This fiasco may be avoided by dovetailing a curve of the Type II. on to an initial tract of Type VII., after the manner shown in Section IV.

VIII. There is a certain affinity between Mr. White's formula and another which has been proposed in the ECONOMIC JOURNAL by Mr. Steggall.[1] Mr. Steggall's scheme presents two distinctive features: (*a*) that between certain limits, *e.g.*, between 100 and 1000 the tax on successive equal increments of income increases by an equal increment, *e.g.*, on the first hundred (exempted), 0; on the second hundred, 2*d.* in the pound; on the third hundred, 4*d.*; and so on. The total paid for ten hundreds will thus be $0 + 2 + 4 + \ldots + 18$, an arithmetical progression of which the sum is 90, and accordingly the *rate* for 1,000 is 9*d.* in the pound. If this rate of progression were to continue, we should reach the rate of a pound in the pound too soon. Accordingly (*b*) it is arranged that the rise of 9*d.* in the rate per pound which occurred in the tract of income from £100 to £1000 should thereafter be spread over a larger tract from £1000 to £10,000. After £10,000 the next rise of 9*d.* is spread over the tract £10,000 to £100,000. And so on. It is this latter arrangement which has some affinity to Mr. White's construction.[2]

The other feature (*b*) of Mr. Steggall's plan is one that frequently appears in popular schemes of taxation. Numerous examples will be found in recent reports on the Income Tax.[3] A particularly good instance is furnished by the Wisconsin system. There the tax on successive increments of 1000 dollars each rises from the first thousand to the fifth by $\frac{1}{4}$ per cent, for each 1000 dollars up to the fifth thousand inclusive; and the result of these rates on successive increments is shown as the "true rate on whole amount" (of taxable income);[4] that is, the rate in the usual sense which has been here all along adopted. There is no essential difference between the "Tariff System,"[5] as it is called in the Report, in which each successive increment is subject to a rate increasing in arithmetical progression and the simpler plan

[1] Vol. XXV. (1915), p. 136, *et seq.*

[2] Observing that the logarithm of the taxable income, x, increases by equal increments as the tax increases by increments of 9*d.*, let us suppose these increments to become indefinitely small; and we obtain the simple relation $X = 10^{bR}$; where R is the rate of taxation ($= T/X$); b is a constant: whence $R = \frac{1}{b} \log X$, which corresponds to Mr. White's formula, *mutatis mutandis*.

[3] See 1906, 365; and 1913 [Cd. 7100], *passim*.

[4] *Loc. cit.* (1913), p. 178. The rise of the rate by steps of half per cent. from the fifth to the thirteenth thousand is similarly treated.

[5] *Loc. cit.*, p. 6.

in which the rate reckoned on the whole taxable quantity increases in an arithmetical progression.[1]

To obtain a continuous curve corresponding to the series of steps presented by such arithmetical progression, let us suppose that in any tract in which there is a uniform progression of the sort the steps become smaller and smaller.[2] In the limit the curve representing the rate of taxation will be a parabola. The common parabola emerges as an *eighth* form, as a candidate for the representation of the *rate* of taxation; and accordingly for the tax and the available surplus, a parabola of the third degree. The construction is only applicable to short tracts; otherwise, the continued increase of the tax would be fatal.

IX. The parabola of higher degrees naturally follows here. The formula is recommended by its common use in many branches of physics. It is not, however, applicable to all branches. It is not adapted, for instance, to represent the extremities of groups of observations. For much the same reason it seems unsuited to represent taxation of the higher incomes and capitals. It has, however, the distinction of being, as far as I know, the highest in the mathematical sense of all formulæ actually adopted in the financial regulations of a great country. In the income-tax adopted by the Commonwealth of Australia,[3] for the tract of income between £2,000 and £6,500, the expression for the tax (in pounds sterling) is :—

$$5333{\cdot}3 - 5x + \frac{12{\cdot}583x^2}{10^3} - \frac{1{\cdot}06}{10^6}x^3 + \frac{0{\cdot}03}{10^9}x^4;$$

where x is the taxable income over and above the abatement which is deducted from the total income. For incomes between £546 and £2000 there is another parabola, one of the third degree. Below £546 the formula is more simply arithmetical. There is an abatement decreasing with the amount of income. If we regard each boundary of a discontinuous tract as impairing

[1] Some relations of the two systems are well exhibited in the *Mathematical Gazette* for May 1916, referring to the Australian Commonwealth income-tax.

[2] Let us suppose that the tract of finite extent a is divided into an indefinitely large number of steps, each measuring Δx. Now at each of these small steps let there be added to the rate of taxation the very small quantity $b\Delta x^2$ (b a finite constant). Then the sum of the arithmetical progression which represents increase of the rate in the tract under consideration is $\frac{1}{2}\frac{a}{\Delta x}\left(\frac{a}{\Delta x} - 1\right) \times b\Delta x^2$; i.e. $\frac{1}{2}ba^2$. Thus the increase of the rate of income is given by a *parabola*, of which a measured from the beginning of the finite tract may be taken as the abscissa.

[3] See Official Year-book of the Commonwealth of Australia for 1901–15; No. 9, 1916, pp. 725–26.

simplicity and mathematical elegance in the same degree as an additional constant, we must pronounce the Australian tax somewhat deficient in that quality; taking into account the number of arbitrary boundaries, as well as of constants proper. In spite of, or rather in consequence of, its mathematical elaboration, the Australian formula has hardly any advantage in respect of continuity over the formless British income-tax.

X. If it is thought desirable to employ more constants than enter into the formulæ of Sections III. and IV.—that is, four excluding the exempted minimum—it is easy to take on an additional tax of the new type after the manner shown in those Sections.

To resume and conclude. Several formulæ old and new have been compared in respect of their use for the purpose of graduating taxes. In this comparison regard has been had to certain general conditions which should be fulfilled so far as practicable and consistent with each other. The conditions taken account of are chiefly (1) that the *functions* employed should be continuous; (2) that they should be familiar; (3) that the amount of taxation should never be so great as to make it the interest of the taxpayer not to increase his income or capital; (4) that the rate of taxation, as the income or capital increases indefinitely, should converge not to a proper fraction (a percentage less than 100), but to unity (100 per cent.); (5) that the abatement which is to be free from taxation on various grounds (children, insurance, etc.) should not enter as a constant into the formula for graduation. To which it is perhaps to be added (6) that some of the constants should be, like the abatement, determinable from considerations of expediency other than their effect upon the result of the calculation, the amount of contribution prescribed by the formula. Comparing proposed schemes, it is not possible to arrange them in an order of merit abstractly, without knowing firstly the end in view—in particular at what points of taxable income (or capital) it is expedient to lighten or tighten taxation—and secondly, the means available—in particular how many constants may be employed. If the graduation required is not very steep, several formulæ may be appropriate which would otherwise become impracticable. If the number of available constants is given, certain hypothetical preferences may be expressed.

If (exclusive of, or in addition to, the exempted minimum) only *one* constant is allowed, the form recommended is

$$T = x - x^b,$$

where T is the tax, x is the taxable income or capital, b is a proper fraction. If there are (besides the abatement, as before) *two* constants, we have a choice between these two expressions :

$$(1)\ T = \frac{rx^2}{M + x} \qquad (2)\ T = xr\,\mathrm{Log}\left(1 + \frac{x}{c}\right).^1$$

Sometimes, if the graduation is not very steep, the latter is preferable; but it may be much worse. If it is advisable to have as many as, and not more than, *three* constants (besides the abatement) there is recommended a combination of two prescriptions,[2] namely, (1) $T_1 = x - x^b$, (2) $T_2 = x - ax^\beta$.

If *four* constants are to be utilised, there may be advised a combination of the two taxes :

$$(1)\ T_1 = \frac{rx^2}{M + x} \qquad (2)\ T_2 = x - ax^\beta.$$

By taking on an additional T, or more than one, any number of constants, odd or even, may analogously be employed.

[1] Ordinary logarithms.

[2] T_1 is to be taken by itself up to a certain point—such a point, for instance, as that at which the super-tax begins in the British system. After that point either T_1 and T_2 are to be compounded, or, perhaps preferably, T_2 is to be employed by itself.