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APPLICATIONS OF PROBABILITIES TO ECONOMICS

[THE Probability of which the application to Economics forms the subject of this article (published in the ECONOMIC JOURNAL, June 1910) is largely of the sort which has been called *a priori*, or "unverified." The controversy with Professor Seligman concerning the incidence of taxation in a regime of monopoly—prolonged by his return to the subject in the third edition of his *Shifting and Incidence*—illustrates the use and need of such Probabilities.

They are employed in the second part of the article to show the advantages of Discrimination, or differentiation of prices, in a regime of Monopoly, and even in one of pure Competition. In this argument I have perhaps practised unnecessary moderation in dwelling somewhat exclusively on the case where the disturbance of price caused by discrimination is inconsiderable. True, in the case where differential prices are compared with the single price of pure competition there is no ground for expecting that the advantageous divergence of prices will be considerable. But in the case of discrimination introduced under a regime of monopoly there is room for considerable differentiations advantageous to both parties. Also the proposition, placed third below (p. 407), that the monopolist may be induced for a small consideration to adopt a system of prices through which the customers, as well as himself, may be materially benefited by the introduction of discrimination in the place of a unique monopoly price—this useful proposition may well hold good of considerable discrimination, in virtue of a property of a maximum to which attention has been often called in these pages (see Index, *sub voce* "*maximum*"). Indeed the verifications of the theory which have been offered are by no means confined to indefinitely small discriminations. The examples observed are such that in each the price of one of the articles is raised by 12½ per cent.]

Probability, in that general sense in which it has been called the guide of life, is of wider application to Political Economy than dogmatic theorists imagine. Of this extensive field one corner only is here explored, namely, that species of Probability which is amenable to mathematical theory. The Calculus of Probabilities and the principles of Economics may seem, indeed, quite distinct branches of knowledge. Yet I submit that they have a root in common. The theory of Probabilities lends to Economics, as to other sciences, certain premises which are evidenced, neither by pure intuition, nor by formal induction, but by general impressions, and what may be called mathematical common sense. Of this character is " the assumption that any probability-constant about which we know nothing in particular is as likely to have one value as another "—an assumption made by Laplace and endorsed by Pearson.[1] This so-called *a priori* probability is employed in the theory of measurement, not only with respect to probability-constants, but also for the determination of other kinds of constants. The coefficients which characterise curves and surfaces and analytical *functions* corresponding to these geometrical forms are presumed, in the absence of evidence to the contrary, not to have extreme values, not to be enormously great or indefinitely small. For where the possible values range over a certain tract with some approach to equality of distribution, it may be presumed that they do not often occur at the limits of that tract. The curves with which we have to deal in practice do not abound in *singularities*. The increment of a function corresponding to an increment of the independent variable is not commonly immense. There usually holds good, or is tenable as a working hypothesis, the fundamental principle that of interdependent variable quantities an increment of one is attended with a (simply) *proportional* increment of the other; throughout a tract of sensible extent.[2]

" Commonly " and " usually " may seem strange terms to occur in a mathematical proposition. But I submit that the conception is required by mathematicians. Thus Hamilton, in explaining the definition of " differentials " (with reference to his

[1] Pearson, *Grammar of Science*, second edition, p. 146.

[2] Beyond a certain tract—a certain amount of change in one of the quantities, considered as the independent variable—it may be expected that the *proportion* itself changes, yet not *per saltum*; it being assumed, when nothing is known to the contrary, that $\frac{d^2y}{dx^2}$, as well as $\frac{dy}{dx}$, is not immense. In short, the Taylorian expansion in ascending powers is presumed to hold good for two or three terms at least. Much use will be made of this presumption in the sequel of this paper.

own original calculus) has to employ the phrase " in all ordinary cases." [1] The following is a typical extract from Laplace.[2] It relates to the problem of *correcting* an " element " [quantity under measurement] already approximately determined, by means of numerous observations each of which represents, or purports to be equal to, a known *function* of the element.

" If we substitute in that function instead of the element its [known] approximate value *plus* the correction [called] z; then develop the function in ascending powers of z,[3] and neglect the square of z [and higher powers thereof], the function will assume the form $h + pz$ [where h and p are constants]."

Now, of course, it may happen that the coefficient p (and accordingly the coefficient of z^2 in the expansion) is enormously great, and so the proposed simplification will not be available. Laplace, however, in the above cited and many other passages, tacitly assumes that this will not happen.

The sort of continuity which must be postulated for practical purposes is not of exactly the ideal continuity about the definition of which mathematicians dispute. A broken line, or succession of dots, may often be treated as if continuous. In Probabilities curves of " error," or " facility," are to be conceived, I hold, as series of discrete points. In Physics the received molecular theory seems to require a similar conception. Thus Clerk-Maxwell, with reference to Atoms :—

" The principle of continuity, which is the basis of the method of fluxions and the whole of modern mathematics, may be applied to the analysis of problems connected with material bodies by assuming them, for the purpose of this analysis, to be homogeneous. . . . Thus if a railway contractor has to make a tunnel through a hill of gravel, and if one cubic yard of gravel is so like another cubic yard that for the purposes of the contract they may be taken as equivalent, then in estimating the work required to remove the gravel from the tunnel he may, without fear of error, make his calculations as if the gravel was a continuous substance. But if a worm has to make his way through the gravel, it makes the greatest difference to him whether he tries to push right against a piece of gravel, or direct his course through one of the intervals between the pieces; to him, therefore, the gravel is by no means a homogeneous and continuous substance." [4]

[1] *Quaternions*, Joly's edition, Vol. I. p. 432.
[2] *Théorie Analytique des Probabilités*, Liv. II. ch. iv. § 20.
[3] So I freely translate " en réduisant en série par rapport à z."
[4] Article on " Atoms," *Encyclopædia Britannica*, ed. 9, Vol. III. p. 38.

I cannot pretend to give an adequate exposition of the philosophy which underlies the science of Mathematical Physics. But if there is a part of that mysterious substructure which at all corresponds to the description above given, then it cannot be considered as paradoxical that a less exact science should rest in part upon similarly inexact axioms.

I submit, therefore, that in Economics we must sometimes be content with premises not better evidenced than those which have been above attributed to Physics. For example, Professor Pigou, reasoning about the incidence of a differential duty on foreign wheat, very properly begins: "Presuming, as in the absence of knowledge is reasonable, that the elasticity of production is the same at home and abroad." [1] If anyone specially conversant with the trade in wheat is able to correct this presumption, that does not prove Professor Pigou wrong in making it, or adhering to it until more exact information is forthcoming. In this world it is often necessary to act though we know only in part. Thus M. Colson, for the important purpose of estimating the utility of a "public work," prescribes that, adopting a mean value [*en moyenne*], in the absence of more exact data, we may evaluate the benefit per unit of new traffic,[2] by a formula which amounts to this, as I understand. Whereas we know nothing of a certain curve, bounding an area which is to be measured, except that the curve joins two given points and that it slopes continually downwards.[3] Under these circumstances we may, for the purpose of the measurement, put for the curve the right line joining the given points.[4] Of course this kind of *a priori* presumption is liable to be superseded by specific evidence as to the shape of the curve; for instance, if there were sometimes ground for supposing it to be convex, as was, in fact, suggested by M. Colson's distinguished predecessor, Dupuit.[5] There is required, I think, in a case of this sort, in order to override the

[1] *Cp.* ECONOMIC JOURNAL, Vol. XIX. p. 105.

[2] *Cours d'Économie Politique*, Liv. VI. p. 203.

[3] By which condition an indefinitely large divergence of the curve from the line is excluded.

[4] In Professor Marshall's construction—more familiar to English readers than M. Colson's (see the review of the *Cours d'Économie Politique* (Liv. I.), III. p. 169)—the sought measure is the mixtilinear figure bounded by the horizontal line representing the increase of traffic, the vertical line representing the decrease of price and the demand-curve; the area PRA in Professor Marshall's Fig. 10, *Principles of Economics*, Book III., ch. vi. § 3, ed. 4, corresponding to an increase of output from OM to OH; an area like $BB''b''_2$ in the diagram representing M. Colson's conception in the review just now mentioned.

[5] *Annales des Ponts et Chaussées*, 1844, Vol. II. p. 367.

a priori probability, either very definite specific evidence, or the consensus of high authorities.

Another application of *a priori* Probabilities to economic curves is made by M. Colson with reference to the probability of neutral equilibrium between demand and supply; the supposition that the equation of demand and supply "might be equally satisfied by every numerical rate which could be supposed," in the phrase of J. S. Mill. Mill has entertained this conception in that central part of his chapter on International Value which has seemed to many the least satisfactory part of the chapter and of the whole work.[1] Of this sort of neutral equilibrium M. Colson says [2] :—

"A coincidence such that two functions should preserve the same value, for all values of the variable extending over an interval not indefinitely small presents a degree of improbability which is equivalent to a mathematical impossibility."

Another example of *a priori* probability is the presumption, in the absence of evidence to the contrary, that demand (and supply) curves will not be of an extreme and limiting form—not very rigid or very inelastic. Unfortified by a general presumption of this sort, we are apt too easily to let pass arguments which take for granted that the demand for some article under consideration, for instance, house-accommodation, is perfectly inelastic.[3]

The issue whether an economic curve slopes rapidly or gently is distinct from the question at what degree of minuteness the continuity of the curve breaks down; what is the *minimum divisible* of currency, the *minimum vendible* of commodity. Thus the question whether there would be a great increase in the consumption of alcoholic liquors, if the heavy taxes now imposed on them were removed, is not much affected by the circumstance that the retail price of a pot of beer can only decrease (say) by a halfpenny at a time. That comparatively trifling circumstance may often be abstracted by the theoretical economist. He cannot always be adjusting his speculative instruments to the two scales of magnitude distinguished in Maxwell's parable of the gravel

[1] It is criticised by Professor Marshall in his unpublished papers on Foreign Trade (the papers referred to in the Preface to his *Principles*), by Professor Bastable in his *International Trade*, p. 11, ed. 3, and by the present writer in the ECONOMIC JOURNAL, Vol. IV. (1894), p. 611. Above, **R**, pp. 1, 23.

[2] *Cours d'Économie Politique*, Liv. I., with reference to the supply of and demand for work: not referring particularly to Mill.

[3] See on this point ECONOMIC JOURNAL, Vol. X. pp. 187–9 (above, **U**, p. 168). *Cp.* above, **S**, p. 71.

hill. He may permissibly devote himself to the difficult calculations proper to the contractor, while he leaves to his critics the easy task of making allowances for the idiosyncrasies of the worm.

Referring for further observations on *a priori* Probability to former publications,[1] I now go on to consider some examples.

I.—Theory of Taxation

For the purpose of securing attention to the obscure subject of this study, it happens, fortunately, that the subject bears upon Professor Seligman's deservedly popular treatise on *Shifting and Incidence of Taxation*. Many of the criticisms with which Professor Seligman has honoured my observations on the theory of taxation in a regime of monopoly turn upon the question whether the assumptions above described are legitimate.

The suggested difficulties about my use of the term "in general," as applicable to the case of continuity, and of "friction" in the contrary case,[2] may, I hope, be removed—or at least reduced to questions of terminology—by the explanation above given. Continuity of the rough sort which characterises curves in Physics and Probabilities may be properly, or at least intelligibly, described as "general" in Economics, affording as it does a first approximation beyond which it is frequently unnecessary to proceed.

Professor Seligman still continues (in his third edition) to ask:

"Is it fair to assume that a small change of price is 'more general' than a great one? And would Professor Edgeworth's elaborate formulæ all hold good if the change of price were substantial?"[3]

To which I reply, as I have already replied (to the second edition):—

"Certainly, the formulæ hold good for substantial changes of price as long as the conditions of a maximum continue to be fulfilled, that is, presumably for some finite distance. . . . Because the mathematical investigation advances by tentative steps it is not precluded from going in the direction of the rise of price as far as any other method [can go safely]."[4]

Cournot's method of short steps is the only (tolerably) safe one; Professor Seligman's giant steps over the space of a quarter

[1] See Index, s.v. *A priori Probabilities*.

[2] *Shifting and Incidence*, third edition, p. 345.

[3] *Ibid.*, third edition, p. 345; second edition, p. 276.

[4] Economic Journal, Vol. IX. (1899), pp. 308, 309; F, p. 163.

of a dollar in his favourite illustration [1] may terminate in a precipice. It is true that not even Cournot's method is perfectly safe. Pure Mathematics—exclusive of applied Probabilities—can only guarantee the safety of a single step; or not even that in case of a *singularity* occurring at the point (of the demand curve) with which we are concerned. In order to obtain a practical conclusion the Calculus of Probabilities must be grafted on the Differential Calculus,[2] as I have elsewhere explained.

Had Professor Seligman entertained the, I think, received assumptions as to the curves employed in applied Mathematics, he would have prevented some misunderstanding of his doctrine as to the relation of elasticity to taxation in a regime of monopoly. Professor Seligman had said (in his first edition) :—" The greater the elasticity of demand the more favourable—other things being equal—will be the position of the consumer." [3] This proposition I very naturally understood to imply that if in the long run of general practice we could distinguish those cases in which the elasticity of the demand-curve was particularly great, we should find that the liability of the consumer to suffer by taxation (in a regime of monopoly) was particularly small. I not only disputed this thesis, but the method of proving it, which seemed to me to consist in the observation of examples not taken at random, but selected as having an attribute favourable to the thesis.[4] It was Voltaire, I think, who said that you could kill a flock of sheep with incantations—if accompanied by arsenic. Now it seemed to me that Professor Seligman, while professing to observe the effects of " incantations," had taken care that " arsenic " should be present in each instance. But on considering his amplified statement (in the third edition) I am led to believe that I was mistaken; that he meant " arsenic " all the time—though he said " incantation." I surmise that, instead of the false thesis above cited, there was intended the true thesis which is obtained by substituting in that statement for " elasticity of demand," *increment* of that elasticity. Or rather we ought to put as that quantity of which the thesis affirms that the greater the said quantity the more favourable will be the position of the consumer, the following, or some equivalent coefficient; the increment, with the increase of price, of the decrement-of-demand-corresponding-to-increment-of-price.[5] For there now comes into

[1] Cited below, p. 395.

[2] ECONOMIC JOURNAL, Vol. XVIII. (1908), p. 399. Above, p. 351.

[3] *Shifting and Incidence*, ed. 1, p. 191; quoted by me in the ECONOMIC JOURNAL, Vol. IX. p. 303; amplified by the author in his third edition, p. 345.

[4] ECONOMIC JOURNAL (*loc. cit.*), p. 304.

[5] The phrase made up of words connected by hyphens is used to denote the

view a distinction which might be safely ignored so long as we were concerned only with a single point on the demand-curve (a point which might, however, be treated as representative of the neighbouring tract, in virtue of the principle just now explained). The distinction which now arises is between elasticity in the *proper* sense of the term as defined by Professor Marshall,[1] and elasticity in a popular sense, presumably the coefficient above defined; or more simply, the slope of the demand-curve. The complications connected with the use of *elasticity proper* were evaded in my former paper,[2] as they will be in the present paper, by the device of (in effect) taking as the unit of commodity and price respectively the values which render the monopolist's revenue a maximum. For the slope and the elasticity of the curve then coincide. But when we pass from the attribute itself to the *increment* thereof, a choice between the two meanings must be made. There can be little doubt, I think, but that our author would choose the popular sense. An intelligible and true meaning can thus be assigned to his statements :

"Demand is said to be more elastic when each successive increase of price leads to a greater falling off in demand." [3]

"After the point of maximum monopoly revenue has been reached, the more elastic the demand the smaller will be the proportion of the tax that he [the monopolist] is apt to shift to the consumer." [4]

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positive quantity $[-F'(p)]$, if with Cournot we put $F(p)$ to denote the amount demanded at the price p. We might, of course, equally well—*pace* Professor Seligman—designate this coefficient as the increment-of-demand-corresponding-to-decrement-of-price. The increment of this coefficient with the increment of price is the greater the smaller that $F''(p)$ is; since by an increment of price, Δp, $[-F'(p)]$ becomes $[-F'(p) - \Delta p F''(p)]$. But the smaller that $F''(p)$ is the less (other things, and in particular $F'(p)$, being the same) is the increase of price due to a small increase of cost of production or taxation, say τ per unit of commodity. For by Cournot's theory (*Principes Mathématiques*, ch. v. § 31) that increase is $\tau \frac{F'(p)}{2F'(p) + pF''(p)}$, which may be written $\tau \frac{[-F'(p)]}{2[-F'(p)] - pF''(p)}$, where the denominator is necessarily positive. The smaller that $F''(p)$ is, the greater is the denominator of the above fraction; and, therefore, the smaller the fraction itself, the less the rise of price in consequence of the tax; in accordance with the thesis enunciated in the text. (It may be well to warn the reader that "smaller," as here applied to $F''(p)$, means that this quantity is less, *account being taken of its sign*. When, as frequently, $F''(p)$ is negative, its value is less the greater its *absolute quantity* is—as a man is poorer the greater his debt is.)

[1] *Principles of Economics*, Book III. ch. iv. Compare Palgrave's Dictionary : article on "Elasticity" (by the present writer).

[2] See ECONOMIC JOURNAL, Vol. IX. p. 291 (F, I. p. 157), and below, p. 428, note. In the sequel we shall have to do with cases in which the elasticity proper is (at assigned points) the same for demand-curves with different slopes.

[3] *Shifting and Incidence*, third edition, p. 346.

[4] *Loc. cit.*, p. 345; more explicit than the corresponding passages in the second edition (p. 277).

It will be apparent that it is impossible even to enunciate the writer's thesis without the use of the methods which he contemns. The above interpretation is put forward with diffidence as it is perfectly consistent with the sort of continuousness which I have all along postulated for demand-curves. Professor Seligman seems to call for a " Deus ex machinâ " where no " dignus vindice nodus " occurs, if my interpretation of his thesis is correct. But the absence from his view of the matter of those conceptions and presumptions which are the subject of this paper is, I fear, fatal to our mutual understanding.[1]

I take to myself blame for the misunderstanding that has occurred, so far as it may have been aggravated by my neglecting to point out the distinction between increment of slope and increment of elasticity proper. But we are not now concerned with ordinary negligence. Such human error is to the defect which we are now considering as ordinary mistakes in spelling are to those which are perpetrated by a type-writing machine. The latter exhibit a want of that minimum of orthography which is common to articulate men. A similar absence of conceptions and presumptions present to the general mind seems to be evinced by our author's persistent use of a certain demand-schedule, with wide intervals between the entries. I once more reproduce in a variant form this scheme of data.[2]

PROFESSOR SELIGMAN'S DATA.

Price in $	4	5	$5\frac{1}{4}$	$5\frac{1}{2}$	$5\frac{3}{4}$	6
Amount demanded	1200	1000	900	825	750	700
Gross receipts ...	4800	5000	4725	4537·5	4312·5	4200
Net receipts A ...	2400	3000	2925	2887·5	2812·5	2800
Net receipts B ...	2100	2750	2700	2681·25	2625	2625

Net receipts A are based on a cost of two dollars per unit of product, without tax.

Net receipts B are based on the same cost, together with a tax of a quarter of a dollar per unit of product.

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[1] The following quotation, with the comment thereon, may illustrate our mutual inaccessibility. " My assumption," says Professor Seligman, " is that of a demand which becomes more or less elastic after the point of maximum monopoly revenue has been reached. Professor Edgeworth's assumption is that of a demand which is more or less elastic from the outset, before, as well as after this point." (*Shifting and Incidence*, p. 347. Note, par. 1.) Upon which I remark: (1) I can accept the assumption which Professor Seligman takes to himself if, as is possible to interpret, it involves no more than the usual presumptions that the slope of a curve and the increment to the slope $\left(\text{say } \frac{dx}{dp}, \text{ and } \frac{d^2x}{dp^2},\right.$ where p is the price and x the quantity demanded at the price) both vary gradually. And (2) I cannot accept the assumption attributed to myself if, as is natural to interpret, it involves something more than those usual presumptions.

[2] See *Shifting and Incidence*, third edition, pp. 343, 344; second edition, pp. 275, 276; and *cp.* ECONOMIC JOURNAL, Vol. IX. pp. 307, 308.

Now if it is merely meant that the transaction is of such a kind that differences of price less than a quarter of a dollar cannot be taken into account, *cadit quæstio*. I have fully admitted that reasoning based on ordinary degrees of continuity does not apply to this particular case.[1] If my critic meant no more than this, he would hardly have repeated (in his new edition) his schedules and arguments. It is incredible, too, that he should regard an exception of this kind as a refutation of Cournot's theory of taxation in a regime of monopoly. One might as well pretend to have damaged the Ricardian theory of taxation in a regime of competition by adducing the well-known little fact that, in the words of a distinguished Chancellor of the Exchequer,[2] a very small additional duty "can hardly fall on the individual consumer of a glass of spirits or a pot of beer." Besides, if this had been our author's meaning, why keep to such wide intervals of price as a quarter of a dollar?

Upon whatever principle constructed, this schedule may be likened to a net with meshes so wide as to lose half the catch. To remedy this defect we might fill up the vacant spaces with a finer reticulation. This will be effected if we put a continuous curve through six points representing the specified amounts of commodity corresponding to the several prices; an appropriate form being assigned to the curve, and the constants being then calculated from the data. Or, as it is rather a troublesome matter to construct such a curve, it must suffice to construct a continuous curve complying with the parts of the data which are essential to the argument.

Here is a particularly simple curve of the sort:—

$$x = 900 + 200\sqrt{5\tfrac{1}{4} - y};$$

where x denotes the quantity of commodity demanded at any price y. Here, in accordance with the data, $x = 1000$ when $y = 5$; and $x = 900$ when $y = 5\frac{1}{4}$. Also xy is, as it ought to be, a *maximum* when $y = 5$; as is proved by general reasoning,[3] and may be verified by actually trying values of y in the neighbourhood of 5, *e.g.*, 4·9 and 5·1.

Now let us suppose, with Professor Seligman,[4] that the cost of production—the tax imposed by the nature of things—is \$2 per unit of commodity. The quantity to be maximised by the

[1] *Loc. cit.*, p. 307.

[2] Vernon Harcourt, Budget Speech, 1894.

[3] When $y = 5$, $\frac{d}{dy}xy = 0$, while $\frac{d^2}{dy^2}xy < 0$.

[4] *Loc. cit.*, p. 343.

monopolist is then $x(y - 2)$. And we cannot suppose with Professor Seligman that the price continues to be \$5; [1] unless, indeed, it is postulated—a postulate surely not of general validity and one to which Professor Seligman has not called special attention—that the monopolist cannot charge a price intermediate between a dollar and a quarter of a dollar. If the formula which we have assigned for the demand-curve were perfectly exact, the phenomenon perfectly continuous, the price would now theoretically be 5·1439 (nearly); that being the value of y which renders the above expression a maximum.[2] The actual price will be an approximation to this ideal limit. If the price can be graduated to a cent, the monopolist will charge, in addition to the original 5 dollars, 14 or 15 cents; whichever of the two makes the net proceeds greater.[3] If, in addition to nature's tax of 2 dollars per piece (for cost of production), there is imposed an ordinary tax of a quarter-dollar per piece, the principle is just the same. We have now to find y so that $x(y - 2\frac{1}{4})$ should be a maximum. The value of y which fulfils this condition is found to be 5·1583. . . .[4] Accordingly, the monopolist will charge an additional 15 or 16 cents, whichever makes the net proceeds greater (no doubt here the latter).

If half-dimes are the lowest admissible denomination, he will charge either 5 dollars and 3 half-dimes, or 5 dollars and 4 half-dimes; if dimes are the lowest denomination, either 5 dollars and 1 dime or 5 dollars and 2 dimes; if quarter-dollars are the lowest admissible denomination, the monopolist will charge either

[1] *Loc. cit.* "He will always prefer the price \$5." Compare the "net receipts A" in our Table, embodying Professor Seligman's data.

[2] We have now to determine y the equation:

$$\frac{d}{dy}(y - 2)(900 + 200\sqrt{5\tfrac{1}{4} - y}) = 0.$$

Whence:

$$900 + 200\sqrt{5\tfrac{1}{4} - y} - \frac{100}{\sqrt{5\tfrac{1}{4} - y}}(y - 2) = 0.$$

Reducing, we obtain a quadratic equation of which the only available root (the other root being negative) is 5·14389 . . .

[3] The integer preferred is not necessarily the one nearest to the theoretical price.

[4] This value is found from a quadratic equation by reasoning of a parity with that employed in the preceding note. It may be well to remind the reader that the reasoning and conclusion are similar in the case of an (*ad valorem*) tax on gross receipts, as distinguished from a (*specific*) tax on gross product. Suppose that an *ad valorem* tax of 5 per cent. is imposed on the gross receipts. The amount which the monopolist will now seek to maximise is $\frac{19}{20}xy - 2x = \frac{19}{20}x(y - 2\frac{2}{19})$; an expression of the same form as that which it was proposed to maximise in the case of a specific tax.

5 dollars, or 5 dollars and 1 quarter-dollar—in each case adopting that alternative which makes the net proceeds a maximum. The last case might be designated by the proposition for which Professor Seligman contends, namely, that "the monopolist will continue to find his greatest profit in continuing to charge the original price"; if we could suppose that an able controversialist in a considered rejoinder would reaffirm as decisive in his own favour a proposition understood by himself and intended to be interpreted in exactly that qualified sense in which it had been explicitly affirmed by the other party to the controversy.[1]

The simple curve which has been adduced is sufficient for the purpose of showing the various results of a tax—all comprehended under one simple law—according as different degrees of continuity prevail. But the illustration is suitable only for taxes of a limited magnitude, as the curve stops short at the price $5\frac{1}{4}$. Here is a more appropriate, though less easily manageable, representation of the proposed law of demand:—

$$x = 842{\cdot}265 + 246{\cdot}227\,(5{\cdot}26289 - y)^{\frac{1}{3}}$$

Here, as before, when $y = 5$, $x = 1000$ (approximately); when $y = 5\frac{1}{4}$, $x = 900$; and xy is a maximum when $y = 5$.[2] Moreover, now when y is greater than $5\frac{1}{4}$ (as well as less than 5), a series of values of x are presented not very different from and corresponding in their general trend to Professor Seligman's data, and equally favourable to his thesis. Thus when the price, $y = 4$, I find[3] that the amount demanded, x, is now $= 1108{\cdot}4$ (instead of 1200); and when $y = 6$, $x = 619{\cdot}8$ (instead of 700). Now let us introduce the cost of production, \$2 per unit piece; and as before we can prove by general reasoning and verify by actual trial that a price intermediate between 5 and $5\frac{1}{4}$—namely, about 5·13—affords the maximum monopoly revenue. Additions to the tax may be exemplified as before, but now on a larger scale.

But, indeed, it is not necessary for the purpose of rendering our author's data typical to adapt thereto an analytical expression, a curve defined by an equation. It suffices to draw with a

[1] See ECONOMIC JOURNAL, Vol. IX. p. 307, where, quoting the proposition again quoted here, I add, "He will" [continue to "charge the original price"] "if he can only alter the price *per saltum* by leaps of $\frac{1}{4}$ dollar. . . . If the monopolist can adopt an intermediate price between \$5 and \$$5\frac{1}{4}$, I submit that he will tend theoretically to do so. . . ." This proposition is entirely in accord with the explanation which has been given by Professor Jannacone (referred to by Professor Seligman) in his "Questione Controverse," first published in the *Riforma Sociale*, xii. (1902).

[2] When $y = 5$, $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} < 0$.

[3] With the aid of logarithms.

free hand a continuous curve passing through points which represent the data as to price and corresponding demand. Or let us take the series of points each of which have for abscissa (measured on the axis OX) the amount of commodity put by the monopolist on the market, and for ordinate that amount multiplied by the price at which that amount is demanded. We may then have a curve in its general shape resembling that in the accompanying diagram; which is copied from an illustration which I employed in a former paper.[1] The revenue which the monopolist seeks to maximise was there compared to the height which a prisoner confined in a narrow vaulted cell seeks to maximise, namely, the vertical distance from the crown of his head to the sole of his foot. That position of greatest comfort is defined by the point P, while the floor OX is horizontal (in the

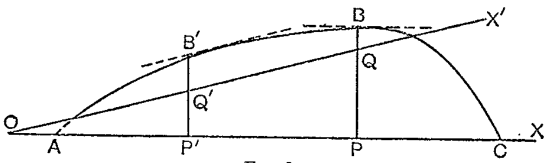

FIG. 1.

absence of taxation, including cost of production). But when (corresponding to the introduction of a tax) the floor is tilted up into the position OX', then the position of greatest comfort recedes from P to P' (the amount put on the market diminishes).

I proceed now to enlarge the cell of the prisoner in order to illustrate a theorem to which Professor Seligman has not, I think, done justice. While he expends labour in proving what comes to nothing but the obvious and admitted fact that when the law of demand is sensibly discontinuous there may occur a suspension *pro tanto* of the general rule that taxation is attended with an increase of price, he passes by slightingly, and fails to see the significance of the theorem that, even though the law of demand be perfectly continuous, taxation in a regime of monopoly may not only not injure, but actually benefit the monopolist's customers—in the case of articles which are to some extent substitutes for each other, such as first-class and third-class accommodation on a railway journey. Reverting to the parable of the prisoner in the cell, we have now to regard the area bounded by the curve $AB'BC$ in the diagram as the section no longer of a

[1] ECONOMIC JOURNAL, Vol. VIII. (1898), p. 236.

narrow cell, but of a vaulted curve which extends far into a third dimension perpendicular to the plane of the paper. Let the reader hold the page before him with its plane perpendicular to that of the table; and let him measure, in a direction from him, along a new axis Y, perpendicular to the plane of the page and parallel to that of the table,[1] a length y representing the amount of a second commodity, say third-class accommodation. Let the amounts of the two commodities (first-class and third-class accommodation) be initially x_0 $(= OP)$ and y_0; and after the imposition of a tax x_1 $(= OP')$ and y_1. Then x_1 is less than x_0. But what about y_1? For all we know the output of the second quantity also may be diminished in consequence of the tax on the first commodity. But this is the less probable alternative; as I have argued from an inspection of the symbols employed.[2] In the case of this the less probable alternative, the phenomenon now in question, the fall of both prices, cannot possibly occur.[3] But in the more probable case of y_1 being greater than y_0 (y_2 greater than y_1, and so on, throughout a tract of sensible magnitude), the concurrent fall of prices may quite possibly occur—however ridiculous it may appear to careless thinkers who transfer to a regime of monopoly the maxims proper to the classical regime of competition. It all depends on the value of certain coefficients which determine the change in prices. The values of these constants which are favourable to the double event are not, I think, *a priori*, extremely improbable. The conditions which they must fulfil * are not excessively exacting. Here is an example which does not strike one as specially unlikely. The number of first-class passengers (per day, week, or other unit of time) being x, and the number of third-class passengers being y, the fares in pounds sterling, p_1 and p_2 respectively, are connected by the equations :[4]

[1] A line parallel to the vertical PB in the diagram may be taken for the axis of Z, on which is measured the amount of the monopolist's net revenue corresponding to assigned outputs, x and y, of the two commodities.

[2] "Teoria pura del Monopolio," *Giornale degli Economisti*, 1897. (E, I. p. 135.)

[3] If not self-evident, this proposition may be deduced from the formulæ (*mutatis mutandis*) in the Note at the end of this paper.

* The conditions are investigated in the Note at the end of this section; in the ECONOMIC JOURNAL (Vol. XX.), not reprinted here. For a popular statement of the *rationale* of the phenomenon see ECONOMIC JOURNAL, Vol. VII. (1897), p. 231 (S, p. 94), and Vol. IX. (1899), p. 288 (F, I. p. 146).

[4] The reader is advised to substitute for $\frac{x}{100{,}000}$ the new variable x, and for $\frac{y}{200{,}000}$ the new variabley.

$$p_1 = \text{L}1\left[\frac{17}{5}\left(\frac{x}{100,000}\right) - 2\left(\frac{x}{100,000}\right)^2 - \frac{2}{5}\left(\frac{y}{200,000}\right)\right]$$

$$p_2 = \text{L}\tfrac{1}{2}\left[\frac{73}{30} - \frac{22}{15}\frac{y}{200,000} + \frac{13}{30}\left(\frac{y}{200,000}\right)^2 - \frac{2}{5}\left(\frac{x}{100,000}\right)\right]$$

The form of these equations is the simplest consistent with (1) the hypothesis that the demand for one commodity is diminished by the possession and consumption of a certain amount of the other commodity, and (2) the presumption that a demand curve is not a right line.[1] As the *form of the functions* presented is ordinary so the *values of the coefficients* do not appear *outré*. And yet a tax on first-class tickets will cause *both* fares to drop.

To show this let us begin with the simple supposition that cost of production (the working expenses assignable to the varying numbers of passengers) may be left out of account. Then the revenue which the company, acting as a monopolist, seeks to render a maximum, is $xp_1 + yp_2$. If p and p_2 are connected with x and y in the manner supposed, this expression will be a maximum, when $x = 100,000$ $y = 200,000$; that is, when $p_1 = £1$, $p_2 = \frac{1}{2}$ £1 (= 10*s.*); as is proved by general reasoning,[2] and may be verified by actual trial of values x and y in the neighbourhood of 100,000 and 200,000 respectively (corresponding to values of p_1 and p_2 in the neighbourhood of £1 and 10*s.* respectively).

Now let a tax of 1*s.* 3 mites per ticket be imposed on first-class passengers. It will then pay the monopolist company to lower first-class fares by about $3\frac{7}{8}$d. and third-class fares by about $3\frac{3}{4}$*d.** For at these new prices the numbers of passengers will be

[1] Unless there occurred the term involving x *squared*, the demand curve for x, which represents the amount of x varying with the price thereof while the amount of y is kept constant, and the corresponding demand curve for y, would degrade to a right line.

[2] Substituting x for $\frac{x}{100,000}$ and y for $\frac{y}{200,000}$, and putting V = 100,000 xp_1 + 200,000 y p_2 we have for the conditions that V should be a maximum, the two (simultaneous) equations :

$$\frac{34}{5}x - 6x^2 - \frac{4}{5}y\left(= \frac{dV}{dx}\right) = 0$$

$$\frac{73}{30} - \frac{44}{15}y + \frac{13}{10}y^2 - \frac{4}{5}x\left(= \frac{dV}{dy}\right) = 0$$

(together with three inequations—relating to the second term of variation). The two equations (and the three inequations) are satisfied by the system of values x = 1, y = 1. Whence $x = 100,000$, $y = 200,000$, $p_1 = 1$, $p_2 = \frac{1}{2}$.

* In the example as now stated there has been substituted for the tax originally considered, viz. 2*s.* per first-class ticket, the smaller figure given above, or more exactly £0·05166. The difficulties attending the use of a strong case for the purpose of illustration are thus lessened. These difficulties were pointed out in a note appended to the original article when it was too late to alter the text. They are summarised below (p. 427).

about 97,880 first-class and just 214,000 third-class. It follows that the gross receipts will be diminished by a matter of £64.* But on the net receipts there will be a gain; for though £64 has been lost, there has been saved about £109, the tax which the monopolist would have had to pay on the 2,120 first-class passengers who have dropped out. There is thus a net saving of about £175.

It is no serious objection that the transaction instanced, passenger fares, is of a kind which does not admit (like some wholesale transactions) of making prices in fractions less than the lowest coin in currency; that a mite ($\frac{1}{8}d.$), and even a farthing, can hardly figure in passenger fares. In such a case, as above fully explained, the monopolist will fix that integer number of the lowest available units (say half-pence)—in the neighbourhood of what we called the "theoretical price"—which affords the maximum revenue. The degree of fineness to which the currency is graduated is a matter of quite secondary interest in relation to a theorem of the kind which we are now considering.

So far we have made abstraction of cost of production. But it may easily be shown that to take it into account is not fatal to the possibility of both prices falling in consequence of a tax. Consider any case of the kind hitherto considered—where there is no cost of production—in which the conditions are favourable to the occurrence of the double event. In such a case it may be presumed, by *a priori* Probability, that as the tax is increased from zero up to some finite amount, say t, the monopolist will continue to subtract more and more from both prices. Now suppose that a part, say half of t, is not an ordinary tax, but that loss of gross receipts which the niggardliness of nature imposes, namely, cost of production. The consequence of imposing a tax (in the ordinary sense) of $\frac{1}{2}t$ per unit (in addition to nature's tax of $\frac{1}{2}t$) will be, under the circumstances, to lower both prices.[1]

* Suppose that y is changed in consequence of the tax from 1 to 1·07 (y from 200,000 to 214,000). Then in order that $\frac{dV}{dy}$ should vanish we must have x = ·978796 (x = 97,880 nearly). In order that $\frac{d}{dx}(V - x\tau)$ should vanish, τ = ·05166. From the new values of x and y we find for the new prices p_1 = £·983824; $2p_2$ = £·9686; each lower than the original figure, namely £1. For the gross profits, $xp_1 + yp_2$, we have now 97879·6 × £·98382 + 107,000 × £·9686 = £199,936. There is thus a loss on the gross profits of some £64. But there is saved on the tax (100,000 − 97,880) £·05166 above £109.

[1] *Cp.* ECONOMIC JOURNAL, Vol. IX. (1899), p. 292 last par. and p. 293. (F, p. 154.)

The improbability of the event in question is, I grant, very considerable; but it is not enormous, does not amount to practical impossibility. The improbability is not even very considerable, I think, if the paradoxical characteristic, benefit to consumers (in consequence of a tax), is defined so as to include, besides the case of both prices falling, the case in which, though one price rises, the " consumers' surplus " is not as usual impaired, but, on the contrary, increased.[1] To secure this result only one condition[2]—not as before, two—must be fulfilled. If Δp_1 and Δp_2 are the increments of the respective prices consequent on an increment of taxation, it is now postulated—not that both Δp_1 and Δp_2 should be negative—but only that $x'\Delta p_1 + y'\Delta p_2$ (where x' and y' are the respective outputs—prior to the tax) should be negative.

The theory in this more general form may seem open to the sarcasm which Professor Seligman directs against the primary form of the theory: that it " will surely be a grateful boon to the perplexed and weary secretaries of the treasury and ministers of finance throughout the world." [3] The suggested discrepancy between common sense and mathematical inference may seem really to be made out when it is inferred that benefit to the consumers as a whole is not a very improbable consequence of a tax. But the appearance of absurdity is obtained by looking only at one aspect of the theory. We have so far confined ourselves to the case in which the output of the untaxed commodity is increased in consequence of the tax. But, if the alternative—quite possible, though *a priori* less probable—case occurs, then the consumer will be damnified in consequence of the tax to an extent beyond what might be supposed. Professor Seligman might with almost equal plausibility have suggested as the outcome of the theory that—whereas it is natural to suppose that the displacement of first-class passengers by the tax will swell the numbers of the second-class—the theory implies that the accommodation of both kinds will be restricted with, of

[1] Professor Seligman's words (employed in another connection) " the more favourable . . . will be the situation of the consumer " (*Shifting and Incidence*, second edition, p. 191, quoted in the ECONOMIC JOURNAL, Vol. IX. p. 303) seem well adapted to describe the increase of Consumers' Surplus; and I had supposed that the words might bear this meaning. But I now find (*Shifting and Incidence*, third edition, p. 348) that Professor Seligman repudiates this interpretation; and I apologise for having supposed that he might have entertained an appropriate conception.

[2] In the symbols before employed the condition requires that $\frac{d^2V}{dxdy}$ should be negative, and Δy positive; the fulfilment of this requirement being necessary but not sufficient.

[3] *Shifting and Incidence*, third edition, p. 214.

course, considerable rise of prices. The "boon" to weary financiers might now be represented as the lesson that it is inexpedient to tax the more expensive species of a commodity, as the consumption of the less expensive species also will thereby be seriously restricted.

But in truth the boon which the theory confers is not a definite rule, but the warning to distrust rules transposed from the regime of competition to that of monopoly.[1]

I hope to adduce, under head II, an example of *a priori* Probability more directly applicable to practice.*

II.—Discrimination of Prices

This second exemplification of applied Probabilities is, like the first, furnished by the theory of Monopoly. The feature of that theory with which we are now concerned is the power of the monopolist to discriminate between different species of commodities and customers, not preserving that unity of price which characterises a perfectly competitive market. The subject may fittingly be introduced by a quotation from the earliest, and still, I think, the highest authority on the theory of discrimination, Dupuit. In his epoch-making paper on the measurement of utility Dupuit puts the following case :—

"Waterworks are constructed for the use of a town situated on a hill which had before great difficulty in procuring water. The value of water had been so high that an annual subscription of 50 francs was required to pay for a daily supply of a hectolitre [22 gallons] . . . . But now that pumps have been set up, that amount of water costs only 30 francs. As a consequence, the consumer will now employ water for less pressing, less essential wants. . . . Again, owing to the improvement of the pumps, or by the mere fact of increased consumption, the price is reduced to 20 francs. Our man will now want to have four hectolitres, so as to be able

[1] Even the terms proper to the regime of competition are to be transplanted with caution; and I agree with Professor Jannacone ("Questione Controverso," *Riforma Sociale*, Vol. XII.) as to the infelicity of the term *Shifting* applied to taxation under the regime of monopoly.

* There is here omitted, not as erroneous, but as elaborated out of proportion to its importance, a note on the probability of a tax on one of two articles which are partially substitutes for each other producing a fall in the prices of both articles, in a regime of monopoly. It suffices to state the conclusion that the phenomenon appears to be quite possible, though far from probable. The note also pointed out that the reasoning must be applied with caution when the tax considered is very heavy.

to clean his house every day. Supply him with water at 10 francs per hectolitre, and he will demand ten hectolitres, so as to be able to water his garden. At 5 francs he will demand twenty hectolitres, to maintain a sheet of ornamental water; at 1 franc he will want a hundred hectolitres, to have a fountain constantly playing." [1]

With reference to this illustration, it may be asked: supposing that water for use within the house and water for external use, in the garden or pond, form two categories between which it is possible for a monopolist to discriminate; is it to be supposed that when the price is lowered from 20 francs to 10 francs, and accordingly water begins to be employed for external uses, the whole of the additional six hectolitres are employed on external uses or part on (additional) internal uses? The question is not explicitly raised by Dupuit; being indeed not relevant to his context. But I am concerned to postulate for the cases of discrimination with which I deal that a lowered price *is* attended with an increased demand for both of the uses. The species of discrimination which I have in view may be made more conspicuous by noticing its absence from another illustration given by Dupuit:—

"A footbridge is constructed between two populous quarters of a large town at a cost of 150,000 francs. At the rate of 5 centimes per passenger the proceeds prove to be only 5,000 francs [per annum]. The concern is accordingly a failure; the entrepreneur who had borrowed the greater part of the 150,000 francs, being unable to pay the interest on this sum, is soon ruined. The bridge is sold to an intelligent man who studies the demand for the use of the bridge, with the object of increasing his own profits. Thus he observes that his bridge connects a quarter of the town in which there are manufacturing works with the quarter in which the workmen live; and that they have, morning and evening, to make a long detour in order to reach their destination. The use of the bridge would greatly shorten the distance which they have to traverse; but a workman could not afford to pay out of his wages as much as ten centimes a day. . . . [Under the circumstances] the proprietor might insert in his tariff a clause to this effect: 'For passengers wearing a cap, blouse, or jacket [2] the toll is reduced to 1 centime.' [He will thus, suppose, gain an additional 3,000 francs from 300,000 new passengers—per working year of 300 days; but he may lose a part of his original profits, 5,000 francs, as] "a certain number of passengers at 5 centimes will, by reason of their attire, benefit by the reduction which was not intended for them."

[1] *Annales des Ponts et Chaussées*, 1844, Vol. II. p. 337.

[2] *En casquette, en blouse ou en veste.*

[However] " by new artifices he may succeed in reducing the loss. Thus he may stipulate that the reduction of the toll shall be given only at the hours at which the workshops open and close, or only to workmen showing a certificate [1] of employment." [2]

In this and other passages Dupuit suggests a type of discrimination which may thus be formulated. Considering the demand for the undiscriminated commodity (*e.g.*, passage of the bridge without respect of persons) as made up of the demands for different species between which discrimination is possible; it is (α) conceived that the demand for one species is independent of, uncorrelated with, the demand for another species—Dives will not offer less because the toll is lowered for certificated workmen; (β) it is admissible, if indeed it is not essential, that the demand for each species is practically limited (*e. g.* the amount of water employed in internal uses will not be materially increased, however low the water-rate falls). A similar conception is entertained by M. Colson, who walks in the way of Dupuit.[3] I recognise that the conception is of great importance for the purposes of both theory and art. But I emphasise it here only to make clear that it is not the case with which I am about to deal. I am indifferent about the attribute (α), and I am not indifferent about (β); I postulate that when price is lowered the amount of each species—as well as of the genus—increases. For example, if there are two species (such as water for internal, and water for external use) whereof the amounts x_1, x_2 are demanded at the prices y_1, y_2, I suppose that (for any assigned value of y_2) x_1 continually increases as y_1 diminishes.[4] The case is quite sufficiently important to reward attention to its properties. In dealing with it, I shall for convenience of enunciation confine my statements mostly to the variety in which only two species are discriminated; but the propositions thus enunicated are readily adapted to any finite number of species.

Concerning the kind of discrimination thus defined, I propose to prove the three following theses :—

1. *Very probably a system of prices can be assigned, such that both the monopolist and his customers may gain by discrimination.* The gain to consumers may possibly be so great that they are better off than they would have been, other things being equal, under a regime of competition.

[1] *Livret.* [2] *Loc. cit.*, 1849, p. 220.

[3] See, for some account of M. Colson's conception, the review of his *Cours*, III., 170; and compare below, p. 421.

[4] Thus in the example designated C at p. 417 below, each of the components (as well as the compound) demands tails off towards infinity as the price sinks to zero.

2. *Probably the prices which the monopolist will fix in order to render his profit a maximum are such that the customers will lose through discrimination;* except when the amount demanded of one species before the discrimination is much less than the amount then demanded of the other.

3. *Probably, if the disturbance of prices caused by discrimination is not considerable, the portion of the monopolist's maximum which is due to the infliction of loss on the customers is inconsiderable.* For a small consideration the (perfectly self-interested) monopolist may be induced to adopt a system of prices such that the customers will not lose through discrimination; for a small addition to that consideration he may be induced to adopt a system of prices such that they will be materially the gainers.

The general presumptions above described as *à priori* are available to show that the first proposition is probable. The gain of the monopolist by discrimination depending on the addition to or subtraction from each price, may be likened to the height, say z, of a surface shaped like a hill, varying with co-ordinates x and y, such as the longitude and latitude of any position on the hill. Now, in general one can reach a higher position on a hill when free to move in any direction than when one is restricted to motion along a certain path. In the case before us a limitation of this sort exists prior to discrimination; the monopolist being constrained to charge only one price for the whole class of commodity, or in other words equal prices for the two species. When this limitation is removed, the monopolist will tend to start off in a direction which has been called the *line of preference;* [1] perpendicular to another line on the plane of xy called the *line of indifference.* Likewise the consumer will have his lines of preference and indifference. But from our general knowledge of the relations between buyer and seller, we may presume that the lines pertaining to one party are not coincident with the lines pertaining to the other party. Accordingly the direction in which both parties can move together (from the original position), both being gainers by discrimination, is probably represented by an angle of sensible magnitude; the probability of mutual gain is measured by the ratio of that angle to four right angles.

The probability thus discerned will appear greater if we formulate what is known about the relation of the monopolist to his customers. On Fig 1 let the addition to, or subtraction from the price of one species be measured from O on the axis OX, OX', and likewise the alteration of the other price on the axis of Y.

[1] *Mathematical Psychics*, p. 22, and context.

Prior to discrimination, the monopolist was constrained to move along a right line, representing the condition that the two prices must be the same, the line TT' making equal angles with the axes. When the monopolist becomes free to move, otherwise than in this line, his line of preference is evidently not in the same quadrant as this line; not in the direction implying that both the variations of price are positive—between OX and OY—nor yet in a direction implying that both variations are negative—between

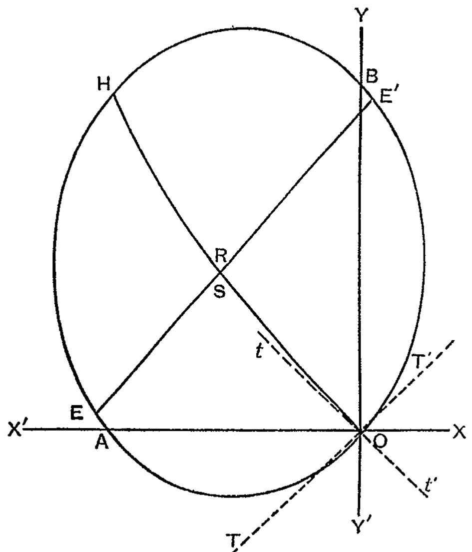

FIG. 2.

OX' and OY'. For if either of these directions represented the monopolist's preference, he would not, prior to the discrimination, have stopped at O. Not his line of preference, but his line of *indifference* slopes in the same general direction as the original path. In the figure the line TT' does duty both for the path of constraint and the line of indifference; but these loci are not in general coincident. But the line of preference pertaining to the customers is evidently in the direction between OX' and OY' since the variation most advantageous to the purchasers is the fall of both prices. Accordingly, their line of indifference will slope in the general direction represented by the line tt' in the figure. The interests of the two parties are concurrent for variations of price which are represented by a step in any direction between OT and Ot.

To obtain an idea of the distance to which they may travel concurrently, we may employ a more elaborate construction; which is also required for the proof of the second and third theses. Let us begin by assigning a particular form to the demand-curves of the customer; and first of all the simplest of all forms, the right line. Let x_1 be the amount of one species of commodity, x_2 that of the other demanded at any price, y; and let $2x = x_1 + x_2$ be the amount of the generic commodity (*e.g.*, water for any purpose) demanded at the price of y. Then by hypothesis x is connected with y by a (linear) relation of the form $x = A - By$; where A and B are numerical coefficients. The monopolist's profit, supposing at first that cost of production may be left out of account, $= xy = Ay - By^2$. This will be a maximum when $y = \frac{1}{2}A \div B$ [1] and accordingly $x = \frac{1}{2}A$. If we call this maximum value of x, a, and the corresponding value of y, b, we have $A = 2a$, $B = a \div b$; and accordingly the equation of the (average, generic) demand-curve may be written in the form

$$\frac{x}{a} = 2 - \frac{y}{b}.$$

This line is represented by BA in Fig. 3; on the supposition that $a = b$ (as may always be effected by properly taking the units of commodity and price).

Let us at first suppose (in accordance with the main portion of thesis 2) that x_1 and x_2 are equal at the price which is fixed by the monopolist prior to discrimination. Let us also for the present suppose that there is no correlation [2] between the demands for the two species of commodity. Then the two specific demand-curves (as they may be called, although they are straight lines) will intersect at the point P, which represents the price and *half* of the quantity demanded before the discrimination. The two curves will diverge at that point as represented by the dotted lines in Fig. 3, in such wise that any horizontal line intercepts between the *average* demand-line (AB) and either of the specific demand-lines (*e.g.* A_1B_1) a length equal to that which it intercepts between the former line (AB) and the other specific demand-line

[1] I use the old-fashioned sign of division $\div$ in the text, but in the more technical notes the now generally adopted sloping line, as thus, A/B.

[2] *Cp.* above, p. 406.

A_2B_2. For instance, on the horizontal line through ω, the intercepts aa_1 and aa_2 are equal. Likewise $AA_1 = AA_2$.

This property may conveniently be represented by the following construction :—

$$\text{Let } x = a(1 + \xi);\ y = b(1 + \eta).$$

Then if ξ and η are measured from P along the rectangular axes, the relation of ξ to η is represented by the line AB (provided that $a = b = 1$). In other words, $\xi = -\eta$.

$$\text{Likewise, if } x_1 = a(1 + \xi_1),\ y_1 = b(1 + \eta_1),$$
$$x_2 = a(1 + \xi_2),\ y_2 = b(1 + \eta_2).$$

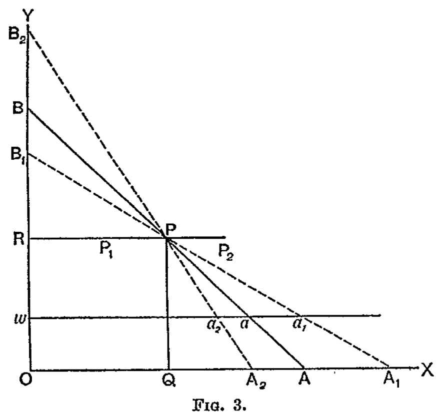

FIG. 3.

$\xi_1 = -q_1\eta_1$, $\xi_2 = -q_2\eta_2$, where q_1 and q_2 must be so selected that $q_1 + q_2 = 2$; say $q_1 = 1 + \beta$, $q_2 = 1 - \beta$, where β is a proper fraction.[1]

The apprehension of the general theorem may be facilitated by assigning a particular numerical value to β. Let us assign a value which is neither very great nor very small, namely (0·2).

$$\text{Thus } \xi_1 = -1{\cdot}2\eta_1;\ \xi_2 = -0{\cdot}8\eta_2.$$

Now the gain of the monopolist through discrimination, say R, being the difference between his profit after the discrimination and what it was before

$$= x_1y_1 + x_2y_2 - 2ab.$$
$$= ab(\eta_1 + \xi_1 + \eta_1\xi_1) + ab(\eta_2 + \xi_2 + \eta_2\xi_2).$$

[1] As follows from the condition that the line representing a demand-curve must be inclined negatively to the axis of x.

Substituting for ξ_1 and ξ_2 their respective values in terms of η_1, η_2 we have (if $a = b = 1$)

$$R = (- 0{\cdot}2\eta_1 - 1{\cdot}2\eta_1^2 + 0{\cdot}2\eta_2 - 0{\cdot}8\eta_2^2).$$

Thus the relation between (changes in) prices, which afford the same profit, the locus of constant revenue, is given by equating the expression within the brackets to a constant. This locus is an ellipse, which when $R = o$ passes through the origin from which ξ and η may be measured (on rectangular axes). In Fig. 1 O represents this origin, and the curve $OAHB$ is supposed to fulfil the condition

$$1{\cdot}2\eta_1^2 + 0{\cdot}8\eta_2^2 + 0{\cdot}2\eta_1 - 0{\cdot}2\eta_2 = O.$$

Likewise the locus of constant Consumers' Surplus is found from first principles to be an ellipse with equation

$$\tfrac{1}{2}1{\cdot}2\eta_1^2 + \tfrac{1}{2}0{\cdot}8\eta_2^2 - \eta_1 - \eta_2 = \textit{constant}.$$

In the figure the curve OSH represents the case in which the said constant is zero, the locus of null gain to the consumers through discrimination.

When the constant in the last written equation is positive, the curve of Consumers' Surplus lies below and to the left of OSH. Consider in particular the curve of this family passing through T, on the supposition that the point T represents the (lower, identical) prices which would prevail, other things being equal, if the regime were one of competition not monopoly. It seems quite possible that this curve (not shown in the figure) should *cut* the locus of null monopoly profit, the ellipse $OAHB$.* There will then be intercepted between these two curves an area any point in which represents a pair of prices which fulfil the secondary part of our first thesis.

The range of variations in price, from O the position before discrimination, that are advantageous both to the monopolist and his customers is represented by the space intercepted between the curves OSH and OAH. The point H may be described as the limit of the range and the index of its extent, if it is understood not to mean that the direct path from O to H can be travelled concurrently, with mutual advantage, by both parties. So the Pillars of Hercules are described by a geographical writer as the limit up to which the navigation of the early Mediterranean peoples extended; though a people situate like the ancient inhabitants of Marseilles could not sail in a straight line to that limit, but must hug a curvilinear shore (that of Spain) comparable with our curve OSH.

* *Cp.* **D**, I. p. 101 *seq.*

The index thus defined is found [1] to be the point of which the abscissa (η_1) is $-$ 0·1855.., and the ordinate (η_2), $+$ 0·2268..; corresponding to prices relatively 18·55 per cent, lower and 22·68 per cent. higher than the prices prior to the discrimination. There is thus a considerable range of variation; considerably greater, as will presently appear, than that which corresponds to the monopolist's maximum profit. Thus the first thesis is verified.

Going on to the second thesis, we have first to determine the prices which render the monopolist's profit a maximum. They prove to be $\eta_1 = -\frac{1}{12}$, $\eta_2 = +\frac{1}{8}$.[2] We have now to observe how the Consumers' Surplus is affected by the adoption of these prices. Substituting the values of η_1 and η_2 in S, the expression for the consumers' gain by the discrimination, we find the gain to be *negative*, namely—$\frac{1}{32}$. The *sign* of this quantity is all that is required to fulfil the second thesis; but it is interesting to notice that the *amount* of loss is greater [3] than the amount of the monopolist's gain, viz., $\frac{1}{48}$ (that is, a gain of about 2 per cent. upon his profits before the discrimination).

To verify the third thesis, we have to compare the maximum monopoly revenue, R', as above determined, the *absolute* maximum as it may be called, with that *relative* maximum, which is the greatest possible gain to the monopolist consistent with the condition that there should be no loss to the consumer. The required positions may be explored by means of the theorem [4] that the maximum monopoly revenue *relative* to, or limited by, the condition that the Consumers' Surplus should have any assigned value is realised by a system of prices such that the elasticity [5] is the same for each of the demand-curves. In the simple case before us, the locus of equal elasticity is a right line inclined to the axis x at an angle of which the tangent is $\frac{27}{2}$,[6] and passing through the point R, the line ERE' in Fig. 2. Thus we have only to determine the intersection of this line with the curve of null gain to the consumers. Let θ_1 and θ_2 be the respective differences between the known co-ordinates of R, η'_1, η'_2, and the sought co-ordinates of the point of inter-

[1] The calculation is facilitated by the incident that the intersection of the two curves is also the intersection of either of them with the line $\eta_1(1 + \frac{1}{2}\beta) + \eta_2(1 - \frac{1}{2}\beta) = 0$.

[2] Differentiating R with respect both to η_1 and η_2, and observing that the second term of variation is negative.

[3] In absolute quantity.

[4] See Note at the end.

[5] The elasticity *proper*, referred to on a preceding page (above, p. 393).

[6] In general $\frac{1+\beta}{1-\beta}\Big/\frac{1+\frac{1}{2}\beta}{1-\frac{1}{2}\beta}$.

section S, say η''_1 and η''_2. Substituting in the expression for S, the Consumers' surplus, for η_1, $-\frac{1}{12}+\theta_1$ and for η_2, $+\frac{1}{8}+\theta_2$, and then putting $\theta_2=\frac{27}{22}\theta_1$, we obtain for θ_1 a quadratic equation of which I find the root to be $-\cdot014067$. The corresponding value for θ_2 is $-0{\cdot}01726$. Whence we obtain for η''_1, $-0{\cdot}09740$ and for η''_2, $+0{\cdot}10774$. Substituting these values for η_1 and η_2 in the general expression for R, I find for the new value of R, R'', as we may call it, 0·02035. This is to be compared with R', the absolute maximum, namely $\frac{1}{48}$, or 0·02083. The difference between R' and R'' is very small, namely, 0·00048; about 2·3 per cent. of R'. That is the proportion of the monopolist's maximum profit which is dependent on the Consumers' loss—a very small proportion in accordance with our third thesis.

When, other things being the same, we suppose the extent of discrimination as measured by the constant β to be increased, it will be observed that the first and the second theses continue to hold good. But the subordination predicated by the third thesis becomes less and less; though it retains some significance for values of β much greater than that which we have considered—say up to $\frac{1}{2}$. To illustrate the failure of the third thesis (while the first and second are eminently fulfilled) put $\beta=1$. Proceeding as before, I find for R' now 0·8, for $R'-R''$, 0·1205; the latter more than 15 per cent. of the former.*

The case which has been considered in which the demand-curves with which we are concerned are straight lines may be regarded as a *Lemma*, which forms a convenient introduction to the far more typical case in which the curves are of the second degree, to wit, parabolas. One obvious difference between the type and the Lemma is the incident that whereas before in the expression for R and S there occurred only *squares* (and first powers) of the variables (η_1 and η_2), there now occur *cubes* of those quantities. But this difference is not from the present point of view the essential one; since the η's are supposed to be so small, or at least so far from great, that their *third* powers may be, I will not say "neglected," but *subordinated*, in comparison with the second powers. It is a more essential circumstance that the coefficient of the second powers in the expression for R now takes on different values, depending on a certain coefficient which is of great significance in the theory of monopoly.[1]

* But see Introduction to this article.

[1] The coefficient ω, as to which see the final Note, p. 427. If the equation to the typical parabola is $\xi=-\eta-\lambda\eta^2$, the coefficient of η^2 in R, viz. $-(1+\lambda)$, $=-\frac{1}{2}\omega$.

Still facilitating the acceptance of general truth by a particular example, let us suppose that the demand-curve for $2x$ ($= x_1 + x_2$) prior to discrimination is a parabola of the kind sometimes called horizontal; so that x is of the form $A - By^2$ (A and B both positive). If as before we express the coefficients in terms of the values of x and y, for which xy is a maximum, we have

$$\frac{x}{a} = \frac{3}{2} - \frac{1}{2}\left(\frac{y}{b}\right)^2.$$

Whence, if as before $x = a(1 + \xi)$, $y = b(1 + \eta)$,

$$\xi = -(\eta + \tfrac{1}{2}\eta^2)$$

$$\xi_1 = -1{\cdot}2(\eta_1 + \tfrac{1}{2}\eta_1^2);\ \xi_2 = -0{\cdot}8(\eta_2 + \tfrac{1}{2}\eta_2^2).$$

Proceeding as before, we shall now find

$$R = -0{\cdot}2\eta_1 - 1{\cdot}8\eta_1^2 - 0{\cdot}6\eta_1^3, + 0{\cdot}2\eta_2 - 1{\cdot}2\eta^2_2 - 0{\cdot}4\eta^3_2$$
$$S = -\eta_1 + 0{\cdot}6\eta_1^2 + 0{\cdot}2\eta_1^3, - \eta_2 + 0{\cdot}4\eta_2^2 + 0{\cdot}13\eta^3_2.$$

The intersection of these curves forms the limit to the range of prices advantageous to both parties. If we leave out of account the terms in R and S which involve *third* powers of the η's, we may proceed as before to find the co-ordinates H_1 and H_2 of the intersection. They are respectively $0{\cdot}1\dot{2}\dot{7}$ and $0{\cdot}14\dot{5}$;[1] of the same order as the true values obtained by taking into account the third powers of the variables, namely 0·1258 and 0·1438 respectively.

The values of H_1 and H_2 prove to be in this instance, as in the Lemma, considerably greater, roughly speaking, about double those of η'_1 and η'_2, the co-ordinates which represent the prices affording maximum profit to the monopolist. For these I find:

By the summary method,

$$\eta'_1 = -0{\cdot}0\dot{5},\ \eta'_2 = +0{\cdot}08\dot{3};$$

Taking account of the subordinate cubic terms,

$$\eta'_1 = -0{\cdot}05719,\ \eta'_2 = 0{\cdot}08012.$$

Whether calculated by the true or the approximate method, the values of R', the monopolist's maximum gain by discrimination, and S', the consequent loss to the customers, prove to be much the same; and accordingly the relation between them not materially different. As thus :—

	R'.	$-S'$	$S' \div R'$
Approximate	0·01388	0·02315	1·667
Accurate ...	0·01378	0·02222	1·612

[1] The calculation of the co-ordinates is facilitated by the circumstance that the point of intersection between the curves lies on the straight line $8\eta_1 + 7\eta_2 = 0$. It happens (in this particular example) that this convenient proposition holds good for the true curves, including the cubic terms, as well as of the curves truncated by the omission of those terms.

The approximate calculation may evidently be trusted as a verification of the second thesis.

Going on to the third thesis, I find approximately after the manner of the Lemma, for the prices which make the monopolist's profit a maximum subject to the conditions that the customer is not a loser (or gainer),

$$\eta_1'' = \eta_1' - 0{\cdot}010044,\ \eta''_2 = \eta'_2 - \tfrac{21}{16} = 0{\cdot}010044$$

where η'_1 and η'_2 have the approximate values above found, namely, $0{\cdot}0\dot{5}$ and $0{\cdot}08\dot{3}$ respectively. Whence it is deducible that the gain which the monopolist must forgo in order not to occasion loss to his customers is about 0·0004, about 3 per cent. of the absolute maximum (above stated).* To compare the true result with this approximate one would require a very laborious calculation. The following partial test must suffice. Assign to the ordinate η_2 a value less than that which affords the (true) maximum profit by an amount which the approximate investigation suggests; for example, put $\eta''_2 = 0{\cdot}07$, less than η'_2 ($= 0{\cdot}08012$) by about 0·01. Now find that abscissa of the curve $S = O$ (roughly as to the general shape of that portion with which we are concerned illustrated by the curve *OSH* in Fig. 2), for which the ordinate is 0·07. That abscissa is found to be − 0·06548. Accordingly, + 0·07 and − 0·06548 represent prices for which the consumers' loss is null. But the gain which the monopolist forgoes by the adoption of those prices, say η''_1, η''_2 instead of η'_1, η'_2, is found (by substituting those values in the expression for R) to be a small percentage of R', namely, about 2 per cent.[1] But that percentage, small as it is, exceeds the true percentage which would be obtained by using the true η''_1 and η''_2 instead of the assumed or "trial" values.

The peculiar interest of this example is that it is typical of an immense variety of demand-curves, or *functions* representing x, the amount demanded in terms of y, the price.[2] Very generally,

* Three per cent. of the addition to his profit due to discrimination; but less than ½ per mille of his total profits.

[1] About ¼ per mille of the monopolist's total profit.

[2] The essence of the general reasoning may be indicated as follows. In the notation above employed we have for R', the gain of the monopolist through discrimination

$$(1 + \xi_1)(1 + \eta_1) - 1,\ + (1 + \xi_2)(1 + \eta_2) - 1;$$
$$= -\beta\eta_1 - (1 + \beta)\tfrac{1}{2}\omega\eta_1^2\ldots,\ + \beta\eta_2 - (1 - \beta)\tfrac{1}{2}\omega\eta_2^2\ldots;$$

the *dots* indicating omission of terms involving higher powers, and ω denoting the coefficient which in general terms, with reference to a monopolised article of which the quantity sold is x, the price y, the gain of the monopolist being $V = \frac{1}{x}\frac{d^2V}{dy^2}$. (See final note, below, p. 426, and remember that the quantity sold in virtue of presumptions above enunciated,[1] such a function may be expanded in ascending powers of y of the type

$$A + By + My^2 + Ny^3 \ldots ,$$

with a coefficient M of such an order of magnitude in comparison with subsequent coefficients that, y being a small fraction, the first three terms of the expansion afford an approximation to the value of the function that is adequate for purposes like the present one. If a thesis like ours, not demanding numerical precision, is true of this approximation to the function, it is probably also roughly true of the function itself.

Of course, it must be presumed that the functions with which we are concerned are of an ordinary character—not discontinuous or otherwise abnormal. For example, suppose one of our demand-curves to have the following extraordinary form. Ascending from Q (at zero price) the locus is a vertical line, say as far as P—it is the perpendicular from P on the axis OX—in Fig. 3. From P the locus is a horizontal line, the perpendicular from P on the axis OY. In this peculiar case our first thesis would be fulfilled; all the better, as

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before discrimination is here taken as unity.) Whence for the prices affording maximum profit we have

$$\eta_1' = -\frac{\beta}{(1+\beta)\omega} \ldots, \quad \eta_2' = \frac{\beta}{(1-\beta)\omega} \ldots$$

Also the gain of the customers by discrimination

$$= -\eta_1 - \int_0^{\eta_1} \xi_1 d\eta_1, \quad -\eta_2 - \int_0^{\eta_2} \xi_2 d\eta_2$$

$$= -\eta_1 + \tfrac{1}{2}(1+\beta)\eta_1^2 \ldots, \quad -\eta_2 + \tfrac{1}{2}(1-\beta)\eta_2^2 \ldots$$

Substituting in this expression for S the above-written values for $\eta'_1\ \eta'_2$, we obtain for S'_1 the gain of the customers through the discrimination

$$-\frac{2\beta^2}{\omega(1-\beta^2)}\left\{1 - \frac{1}{2}\frac{1}{\omega}\right\}$$

Which will be negative in accordance with the second thesis, unless ω is small, $< \frac{1}{2}$.

To prove the third thesis consider R and S as functions of θ_1, θ_2, where $\theta_1 = \eta_1 - \eta'_1$, $\theta_2 = \eta_2 - \eta'_2$. Then the position of relative maximum as above defined must lie on the locus of common tangents to curves of the respective families R = *const.*, S = *const.*; that is

$$\frac{dR}{d\theta_1} \Big/ \frac{dR}{d\theta_2} = \frac{dS}{d\theta_1} \Big/ \frac{dS}{d\theta_2}.$$

Whence we obtain (R not involving the first powers of the θ's) $\theta_2 = q\theta_1 \cdots$, where q is a coefficient of the order unity. Substitute this value of θ_2 for θ_1 in the equation to zero of S, which is of the form

$$S' - A\theta_1 - B\theta_2 \cdots,$$

where S' is of the order β^2, A and B are of the order unity; we find the required value of θ_1 and therefore θ_2 to be of the order of β^2. But R is of the form $R' - (1+\beta)\omega\theta_1^2 - (1-\beta)\omega\theta_2^2 \cdots$ Therefore $R' - R''$ (the difference between the absolute and the relative maximum profit) is of the order β raised to the *fourth* power.

[1] Above, p. 389.

there is avoided all *dead loss—perte sèche* in M. Colson's phrase—that is loss to the consumers, which is not gain to the monopolist. Also our second thesis would be eminently fulfilled; for it would be in the power of the monopolist now to charge prices (b_1 and b_2) by which not only one group of customers, but both groups, would have a bad bargain: Consumers' Surplus being theoretically zero or practically only just above it. But our third thesis in this peculiar case would fail altogether. Peculiar as it may seem, this example is not essentially different from one which is at least suggested by very high authority—the Dupuit-Colson type referred to on a previous page,[1] if the attributes there designated a and β are supposed predicable in their strictest form. We are presented with the conception of the area within the demand-curve resolvable into a series of separate columns—as it were so many sacks standing upright, each of which the monopolist can deplete down to any point which it pleases him to fix.[2]

To return to probable matter, if the discrimination is not so complete as to suspend the ordinary properties of demand-curves the theory above propounded may be considered as evident *a priori* in our sense of the term. Accordingly it does not stand in need of specific verification. Nevertheless, as even in mathematics seeing is believing, as the temperament of Didymus is prevalent among those whom I wish to persuade, I have thought it worth while to verify my theory by showing that it holds good for several different laws of demand. For this purpose I select four functions which are in very common use throughout applied mathematics.[3]

[1] Above, p. 404.

[2] As I interpret, there is supposed to be reached a stage of analysis at which the ordinary properties of a demand-curve break down; much as the soap-bubble breaks when the tenuity of the film approaches the dimensions of the constituent molecules. The distinguished economists who entertain this conception are aware of the impossibility of perfectly realising it in practice (*cp.* Dupuit, *Annales des Ponts et Chaussées*, 1842, Vol. I. p. 222; Colson, *Cours d'Économie Politique*, Vol. VI. p. 38. *Cp.* p. 227, par. 2).

[3] The following table exhibits the functions which are employed in two forms;

	x.	ξ.	$\frac{1}{2}\omega$.
A	$\frac{1}{4}(3 - y)^2$	$-\eta + \frac{1}{4}\eta^2$	$\frac{3}{4}$
B	$\sqrt{3 - 2y}$	$\sqrt{1 - 2\eta} - 1$	$\frac{3}{2}$
C	$-\log y/e$	$-\log(1 + \eta)$	$\frac{1}{2}$
D	e^{1-y}	$e^{-\eta} - 1$	$\frac{1}{2}$

the first referred to the zero of commodity and the zero of price as origin, and

There is first (A) the one most used and most useful of all, to evaluate which requires only the operations of arithmetic up to and including *Involution;* in short, the parabola—the common parabola, if no higher power than the second occurs. An example of this law has already been given. But it may be well to consider a second example of a variety less favourable to our third thesis. Next (B) we shall place a function which requires *Evolution* so far as the extraction of the *square root.* Next comes (C) a function which is of wide application in physics, and even in economics has been frequently employed,[1] the *logarithm.* Then follows (D) the nearly related function, which is sometimes called the *anti-logarithm.*[2] I have experimented on these functions in the following uniform manner. I take a curve of the kind under consideration to represent the *average* law of demand, the *half* of the amount demanded at any assigned price, of both species of the commodity. To represent the demands separated by discrimination, I suppose this curve to be thus disturbed, or strained. To half the quantity of x demanded at any price, y, there is added the quantity $\beta\ (x - a)$ to constitute x_1, the demand for one species at that price; and from the quantity of x there is subtracted the quantity $\beta\ (x - a)$ to constitute x_2; where, as before, β is a (not large) proper fraction, a is the amount of commodity sold and b the price which affords maximum profit to the monopolist prior to discrimination. (The enunciation applies primarily to the tract of curve for which x is larger than a; for the tract below that point we may read $a - x$ for $x - a$, and interchange the words "addition" and "subtraction.") The fraction β is in each case determined so as to render the increase of the price that is raised equal to $12\frac{1}{2}$ per cent. of the original price.[3] I now determine an

abbreviated by putting x for x/a, where x is any amount of commodity and a is that amount of which the sale affords maximum profit to the monopolist, and likewise putting y for y/b (*cp.* above, p. 209, par. 2). For the secondary form of the functions the point of which the co-ordinates are $\mathrm{x} = 1$ $\mathrm{y} = 1$ is taken as the origin and the co-ordinates are respectively

$$\xi \equiv \mathrm{x} - 1 (\equiv (x - a)/a)$$
$$\eta \equiv y - 1 (\equiv (y - b)/b).$$

There is added in a third column a coefficient corresponding to M in the immediate context (to ω in the final Note), a coefficient which must be positive and is presumably not a very small fraction.

[1] To represent the law of diminishing returns and the law of diminishing utility.

[2] The inverse of the Napierian logarithm (of the ordinary logarithm divided by the constant 0·434. . .)

[3] It might have been somewhat more elegant, but it would have been considerably more troublesome, to *assign* the coefficient β (as in the treatment of the Lemma) and thence *compute both* the changes of price.

index of the range of prices that are mutually advantageous—those Pillars of Hercules, up to which, as explained with reference to our Lemma, the two parties can travel concurrently. Only it is not always convenient to find the actual position of the Straits; it suffices to find a point, as it were, on the African shore, as in example A, or even as in the other examples, a rock at some distance from that shore, on the Mediterranean side of the Straits. The limits so understood are given in the first column of the subjoined table. I then determine the prices which make the monopolist's profit a maximum, the (money-measure of) loss to the customers by the adoption of those prices, and compare the amount of that loss to the amount of the monopolist's profit when maximised. The percentage given by that comparison forms the entry in the second column. Further, I find a pair of prices which, while not causing any loss to the customers, yet require the monopolist to forgo only a small proportion of the addition (through discrimination) to his (possible maximum) profit; and accordingly a *very* small proportion of his total possible profit. The amount thus forgone, as a percentage of the total profit obtained by discrimination, forms the entry in the third column.

Though I have expended much labour on these calculations, yet, as they are long and delicate, I can hardly hope to have entirely avoided mistakes. Especially the decimals in the Table here following and the final and penultimate places of the decimal in the Table of Materials given in the Notes, are open to suspicion. But I am sure that the computation is quite accurate enough to verify propositions in Probabilities.

TABLE [1] SHOWING VERIFICATIONS OF THE THREE THESES :—

Law of Demand.	1 −	1 +	2	3
A	18·5	22·3	201	2·5
B	18	22	98	2·5
C	18	22	200·5	1·15
D	20	24	309	1·7

(1) Changes of price advantageous to both parties; per cent. of the price before discrimination.

[1] The subjoined table presents the materials from which the table in the text is constructed, namely—

(2) Loss to the customers by discrimination when the monopolist's gain thereby is a maximum; per cent. of the monopolist's maximum profit.

(3) Percentage of maximum profit resigned by the monopolist to avoid loss to the customers by discrimination.*

Is this multiplication of tests like using several triangles of different shapes in order to prove one of Euclid's propositions relating to triangles in general? Or, rather, have we made a contribution towards ascertaining by induction, less roughly than is given by *a priori* evidence, a limit up to which for purposes like ours fractions may be treated as *small*? [1]

β, the coefficient of discrimination;

$(-H_1, +H_2)$ changes of price advantageous to both parties;

$(-\eta'_1, \eta'_2)$ prices rendering the monopolist's gain by discrimination a maximum;

R', the monopolist's gain by discrimination when a maximum;

$-S'$, the loss to the customers by discrimination when the monopolist's profit is a maximum;

$(-\eta''_1, \eta''_2)$, prices in the neighbourhood of $(-\eta'_1, \eta'_2)$ at which the customers are neither gainers nor losers;

R'', the monopolist's gain by discrimination when the prices are $(-\eta''_1, \eta''_2)$.

The prices are relative to the prices before discrimination; the gains (and losses) are relative to the monopolist's profit before discrimination.

Designation of function.	β.	$-H_1$.	H_2.	$-\eta'_1$.	η_2.	$-S'$.	R'.	$-\eta''_1$.	η''_2.	R''.
A	0·140502	0·1856	0·2228	0·08324	0·125	0·03109	0·01547	0·0952	0·105	0·01508
B	0·30217	0·18	0·22	0·08356	0·125	0·03139	0·03213	0·095	0·105	0·03134
C	0·0953109	0·18	0·22	0·090018	0·125	0·0227	0·01132	0·099	0·11	0·01119
D	0·099383	0·2	0·24	0·08316	0·125	0·03117	0·01007	0·1032	0·115	0·0099

The table in the text is thus formed out of the materials.

Column 1 shows H_1 and H_2 each multiplied by 100.

Column 2 shows $-S'/R'$, multiplied by 100.

Column 3 shows $(R' - R'')/R'$, multiplied by 100.

* There should be added a fourth column presenting the more interesting set of percentages which compare the profit resigned by the monopolist (to avoid loss to the customers, viz. $R' - R''$) with the total profit which he might have obtained; viz. $1 + R'$ (*Cp.* Introduction to this article, above, p. 387). These percentages are for A, B, C, D, respectively, 0·04, 0·08, 0·02, 0·01.

[1] Compare Mr. Bickerdike's observation with reference to his theory of "incipient taxes" (ECONOMIC JOURNAL, 1907, p. 101), "Rather strong assumptions have to be made as to the elasticity of foreign demand and supply if the rate of the tax affording maximum advantage is to come below ten per cent."

As I understand (*cp.* ECONOMIC JOURNAL, XVIII. p. 399 *et seq.*), the quantity with which the writer is concerned, the net gain to the home country, consequent upon a small customs-duty, takes the form $Lx - Mx^2 \cdots$; where L is proportionate to the amount of commodity taxed, x is the rate of taxation per unit of commodity; M is such a coefficient as the M described in our text. Or as L must be considered

Having secured this central position, we can now easily extend the territory subject to our laws; removing limitations by which it has hitherto been circumscribed.

So far we have supposed that prior to discrimination the two categories of consumers were equally important to the monopolist, the amount demanded by each at the single price being the same. Now let us recall this assumption; and, beginning with the Lemma, suppose that at the price b ($= PQ$ in Fig. 3) the amount demanded by one group of consumers is $a(1 + a)$, while the amount of the other species demanded is a $(1 - a)$. The first demand corresponds to RP_2 in the figure if a is positive, the second to RP_1 ($P_1P = PP_2 = aa = a$, if $a = 1$). If the specific demand-curves consisted respectively of the lines joining B to P_1 and P_2 (and produced) there would be no discrimination; the two new prices would be identical with the old price, b. But we are to suppose that the dotted lines—not now passing both through P, but one through P_1, another through P_2—are so inclined as to cause a dissilience of prices when the constraining condition that there should be only one price for the whole class is removed. Is it now probable that the consumers as a whole will suffer by the monopolist's using his power of discrimination so as to make his profits a maximum?

Common-sense will perhaps prejudge this question; pointing to instances in which a railway manager may afford a special rate to exceptional classes of travellers (excursionists and so forth). If the general scale of rates is not disturbed by the favour granted to the occasional passengers, if the one species is advantaged and the other is not affected, there must result advantage to the class as a whole.

Doubtless, I reply, in the extreme case of inequality where the demand of the class favoured by discrimination was so small prior to the discrimination as not sensibly to affect the rates fixed for other classes; for instance, the demand of the workmen for the use of the foot-bridge in the second of the illustrations above cited from Dupuit.[1] But we are not now considering extreme cases, but cases in which a—the measure of inequality—is a proper fraction and primarily at least a small one. For instance in the first of Dupuit's illustrations, suppose (what was, perhaps,

as varying with $x =$ say $L' - Nx \cdots$, we may write the quantity under consideration $L' - M'x^2 (M' = M + N)$. The value for which this expression is a *maximum* (approximately $\frac{1}{2}L'/M'$) is probably much smaller than the limit up to which the expression is positive.

[1] *Cp.* above, p. 405.

not his meaning) that of the ten hectolitres of water which are demanded when the (single, undiscriminated) price is 10 francs per hectolitre per annum, six are required for *internal* use and four for *external* use; [1] and that both demands expand when price falls. In such a case are the consumers as a whole likely to suffer by discrimination? The answer given by mathematics to a question in the theory of Monopoly is often not that which is expected by common-sense.

As before, let us put ξ_1, ξ_2 for the proportional or relative changes in demand respectively consequent on the relative changes of price η_1 and η_2. Then we may write

$$\xi_1 = -\alpha - (1 - \alpha + \beta)\eta_1,$$
$$\xi_2 = +\alpha - (1 + \alpha - \beta)\eta_1;$$

simpliciter in the case of the Lemma, or with the addition of terms involving second powers of the η's to fit the more general type. Forming the general expressions for the Monopoly Revenue and the Consumers' Surplus we find that, as long as α and β remain small fractions, the triple thesis is fulfilled nearly as well as when we dealt with β only. Now, likewise, as either of the coefficients becomes large, the second thesis, that the monopolist tends to fix a set of prices prejudicial to the customers, ceases to be qualified by the third thesis, that his interest in their detriment is small.[2] The second hypothesis retains some probability even when the coefficients are considerable; in the absence of knowledge that the forms of the functions with which we have to deal —the higher powers of the variables which now come into play— are unfavourable to the thesis. We are, of course, here, as throughout, contemplating the money-measure of Consumers' Surplus; not taking into account that the consumers on a small scale may be the poor and needy.

If it is given that α is very large (nearly unity) then the exception [3] enounced in connection with Thesis 2 occurs. But if nothing is given about the coefficients, then we may still affirm the thesis in a certain *a priori* sense.[4] No doubt this kind of probability is not so useful as that which obtains when it is given that conditions favourable to the theses, such as the smallness of both α and β, are realised in the particular case with which we have to deal.

[1] Above, p. 405. [2] See observations on the Lemma above, p. 413, par. 3.

[3] This exception deserved to be specified on account of its importance in practice; the attribute by which it is defined—the ratio between the amounts demanded before discrimination—being commonly capable of identification. Theoretically, other exceptions have a right to be enounced; for instance, the case when β is (known to be) large, or γ (below, note 1), or ω small (final Note).

[4] Index, s.v. *A priori Probabilities.*

These considerations are readily extended to the general case.[1]

A further extension of our laws is effected by removing the condition that the commodities in which the monopolist deals should be, like the mineral waters in Cournot's classical illustration, unaffected by cost of production. First, let us make the simplest supposition, that there is a uniform cost of production for all articles of the class considered without regard to the species into which it may be discriminated, or to the total amount produced. This simple case may be represented by measuring in Fig. 2 the net price on which the monopolist's profits are calculated, no longer from the abscissa, but from a horizontal line at a distance from the abscissa, say $O\omega$, which represents the cost of production per unit.[2] The position of maximum profit (prior to discrimination) will now be given by bisecting ωa and ωB, instead of OA and OB. The units of the system being the same as before, the price and amount will not now be each unity. Or if we take the new price and the new amount as the units (in which lengths on the axes are respectively measured), the demand-curve referred to the new position of maximum as origin is no longer $\xi = -\eta$, but $\xi = -q\eta$, where q is a coefficient greater than unity.[3] The essential character of the reasoning is not altered by the modification of the data. Nor is that character altered when, instead of $k_1x_1 + k_2x_2$, representing the total cost of producing the quantities of the specific commodities x_1 and x_2, at the respective rates per unit k_1 and k_2, we have to add a term such as $\pm l_1x_1^2 \pm l_2x_2^2$;[4] where the positive sign corresponds to the *law of diminishing returns*, the negative sign to the *law of increasing returns*; nor when we add a term such as $-l_{12}xy$,[5] corresponding to *joint cost*.[6]

The reader will observe what a subsidiary rôle is here assigned

[1] In general there may be any number of coefficients of discrimination in addition to the α and β which have been introduced. Prior to discrimination let

$$\xi = x/(a-1) = -\eta + \lambda\eta^2 + \mu\eta^3 + \nu\eta^4 \cdots$$

After discrimination

$$\xi_1 = \pm\alpha - (1 \pm \beta)\eta_1 + (1 \pm \gamma)\lambda\eta_1^2 + (1 \pm \delta)\mu\eta_1^3 \cdots;$$
$$\xi_2 = \mp\alpha - (1 \mp \beta)\eta_2 + (1 \mp \gamma)\lambda\eta_2^2 + (1 \mp \delta)\mu\eta_2^3 \cdots.$$

[2] M. Colson employs largely an equivalent construction.

[3] If k is the cost per unit in the new notation according to which the value of x and the value of y which afford a maximum under the new circumstances are now taken as units; then q may be deduced from the condition that $(1+\xi)(1+\eta)-k(1+\xi)$, $= (1-k) + \eta - q\eta - q\eta^2 + kq\eta$, should be a maximum when $\xi = 0$, $\eta = 0$. Whence it is deducible that $q = 1/(1-k)$.

[4] The "l's," as well as the "k's," being positive.

[5] l_{12} being positive. The proposition is, of course, equally true when this coefficient is negative; that is, in the less frequently specified case of rival production (see Index, s.v. *Correlation*).

[6] Nor when higher powers of the variables occur; with the usual assumptions as to the magnitude of their coefficients.

to *joint cost;* which some distinguished writers on Railway Economics seem to emphasise as the principal cause of discrimination. Joint cost is no doubt favourable to discrimination; but there is a more essential condition, unity of management, monopoly.[1]

A further extension is effected by removing the condition that the specific demands should be uncorrelated. The character of the reasoning is not essentially altered by this alteration in the data. The principal difference in the result may thus be expressed. Whereas previously the amount of profits which the monopolist must forgo in order that the customers should not lose or should even gain by discrimination was (approximately) a quantity of the form $A\theta_1^2 + B\theta_2^2$, where A and B are coefficients of the order unity (roughly speaking), θ_1 and θ_2 are of the order β^2 (β being a small, or rather not large, fraction); now there is added to this expression a new term of the same order as the others, $C\theta_1\theta_2$.

I need not point out in detail that most of the propositions above predicated of the Lemma and the simple type are true of the generalised conditions. Enough has been said to show that these propositions hold good through a wide range of circumstances, with as much truth as can be expected of a theory which belongs at once to Mathematical Economics and to the Calculus of Probabilities. Indeed, I am disposed to claim for the theory a greater degree of practical importance than can generally be ascribed to those branches of study.

Mathematical economics serve generally to present comprehensive views as to the interdependence of variable quantities, rather than to solve particular problems; as Professor Pareto has recently pointed out in this Journal.[2] But I submit that there is an exception to this general limitation; that mathematics play a more direct part in the theory of monopoly. What if an exception should be formed by the application of the preceding theorems to one of the doctrines propounded by Professor Pareto himself—not certainly a particular problem, yet a general view which

[1] It may be objected that discrimination arises without monopoly in the case of large establishments; for instance, when an hotel keeper discriminates between wines of different species, though his profits are subject to competition with other hotel keepers. But I submit that he can practise discrimination just because he enjoys a certain degree of monopoly. If the wines were sold separately by open competition, if there was on the spot a sherry-market and a port-market, the prices paid by the customers would each of them—instead of as now on an average, summed up in the hotel-bill—conform to the cost of production. (See Index, s.v. *Quasi-monopoly*.)

[2] In his appreciative tribute to the memory of Walras, March, 1910.

purports to be of direct practical significance? I refer to his argument directed against Socialism, that at best it would not essentially alter the distribution and production of wealth. "Economic goods will be distributed according to the rules which we have discovered in studying a regime of competition. . . ." "Prices reappear," or "will at most change their name."[1] But we have seen that a regulated discrimination of prices, such as might conceivably be practised by a Socialist Directory, but is not possible in a regime of competition, tends to increase the sum-total of utility. A conception still less familiar to popular Socialism is suggested by what may be called the *external case* of our theory, that which is presented when "monopolist" is interpreted to mean sole *buyer*. The suggestion is that to discriminate between labourers on grounds other than efficiency—not always to pay the same wages for the same amount of work done—might diminish the "dead loss" of Producers' Surplus which the contrary policy involves.

But if this advantage is either of a negligible order in relation to the stupendous consequences of a Socialist revolution, or is over-balanced by the liability to enormous abuses; may we not hope for a less precarious application to a more familiar kind of monopoly, the control of railways and generally public works?[2] That hope is justified by experience. For the mathematical principles on which our reasoning is mainly based are actually applied under the skilful direction of M. Colson to the railway policy of France. Such is the proposition that a small reduction of price, so small as to cause a very small sacrifice of profit to the monopolist, is likely to be attended with considerable relief to the customers. Our third thesis but superadds to this received proposition the following one:—In the case of discrimination (in certain not unusual circumstances) the relief to the customers afforded by a small sacrifice of the monopolist's profits is likely to be so considerable that they may be gainers, or at least not losers, by the introduction of discrimination.*

It is true that these propositions are but probable; liable to failure in particular cases. But we are not altogether dependent on the more precarious kind of *a priori* probability, that which is exemplified by the predication of our second thesis[3] in the absence of data as to the extent and elasticity of demand. Such

[1] *Cours d'Économie Politique*, p. 1014 and context.
[2] In the sense of the term in which it is employed by M. Colson.
* See D, and Introduction thereto.
[3] Above, p. 407.

data would often be available sufficiently to show what case we had to deal with. The sun of full knowledge may illuminate part of our course. There may be enough of that daylight to enable us at least to select the proper path; which may then be pursued in safety by the starlight of Probabilities.

NOTE.—*On certain coefficients.* The first differential coefficient of a monopolist's profit with respect to price has an interesting relation to the elasticity of his customers' demand. The former coefficient may be written, in our notation, when there is no cost of production,

$$x + y\frac{dx}{dy} = x\left(1 + \frac{y}{x}\frac{dx}{dy}\right) = x(1 - e);$$

where e is the elasticity as defined by Professor Marshall (*Principles of Economics*). When cost of production is constant, c per unit, the expression becomes

$$x(1 - \epsilon), \text{ if } \epsilon = -\frac{y'}{x}\frac{dx}{dy'}, \; y' = y - c.$$

This proposition may be employed to prove the theorem above enounced (p. 412), that when a monopolist discriminates between different species of custom, subject to the conditions that the subtraction from (or addition to) the benefit of his customers as a whole should be nil, or have any other assigned value, the elasticity of demand is the same for the different species which are discriminated (cost of production being null or constant). For consider the Consumers' Surplus, say W, as the difference between the money-measure of the utility resulting from the consumption of the commodities, and the purchase-money thereof, we have, in the case of two species of commodity,

$$W = \int_0^{x_1} y_1 dx_1 + \int_0^{x_2} y_2 dx_2 - x_1 y_1 - x_2 y_2 = + \int_{\infty}^{y_1} x_1 dy_1 + \int_{\infty}^{y_2} x_2 dy_2.$$

Likewise the profit of the monopolist, say V, is, in the absence of cost of production, $x_1 y_1 + x_2 y_2$. Now the quantity which the monopolist aims at maximising is $V + \lambda W$; where λ is the indeterminate co-efficient proper to problems of relative maximum. We have accordingly

$$\frac{d}{dy_1}(V + \lambda W) = o; \; \frac{d}{dy_2}(V + \lambda W) = o;$$

Whence $\lambda = \frac{dV}{dy_1} \Big/ \frac{dW}{dy_1}$. Now $\frac{dV}{dy_1} = x_1(1 - e_1)$, and $\frac{dW}{dy_1} = x_1$. Therefore $\lambda = (1 - e_1)$. By parity of reasoning $\lambda = (1 - \epsilon_2)$. Whence $e_1 = 1 - \lambda = e_2$. As above, e is replaced by ϵ when the cost of production per unit is a constant, but the proposition loses its simplicity when the cost (per unit) involves the variables. It may be remarked that the property of equal elasticities is also characteristic of another kind of discrimination which may seem particularly suitable for a State Monopoly to practise, namely, that regulation of prices which has for its object the maximum benefit to the purchasers as a whole, consistent with the retention by the monopolist of a fixed profit—a fixed amount, or a fixed percentage of the output, that is, of the cost of production, supposing cost to be constant.

The affinity between elasticity and the increment of monopoly profits extends to the second order of differentials. Putting $V = xy$ we have $\frac{1}{x}\frac{dV}{dy}$ (or is it more elegant to write $\frac{y}{V}\frac{dV}{dy}$?) $= 1-e$ (for any value of the variable). Accordingly, $\frac{1}{x}\frac{d^2V}{dy^2} = -\frac{de}{dy}$, at the point of maximum, since then $\frac{dV}{dy} = 0$. This coefficient, or rather its negative, namely $\frac{de}{dy}$, which has been called ω (above, p. 415), plays an important rôle in the theory of monopoly. It is closely related to, though not identical with, the coefficient designated ω in the note to the first section of this article in the ECONOMIC JOURNAL, Vol. XX. p. 301 (not reprinted here), ω there denoting $\frac{1}{y}\frac{d^2V}{dx^2}$. (*Mutatis mutandis* when cost of production enters.) The coefficient ω is necessarily positive and presumably not (often) very small. The smaller it is, the sooner, as we continue to increase the degree of discrimination, the extent to which prices are varied, is the limit reached at which our third thesis breaks down. Thus in the second of the two parabolas above instanced, a smaller value of η'_2 (β and $-\eta'_1$) will cause the third term of the expansion to become comparable with the second in the case of the second parabola, for which $\omega = \frac{3}{2}$, than in the case of the first parabola, for which $\omega = 3$.

[There is here omitted a paragraph referring to the theorem that a tax on one of two monopolised articles for which the demand is correlated may result in the *fall* of *both* prices. Some difficulties which the reasoning may present when the tax instanced is very heavy, restated in that paragraph, may here be summarised as follows. When the changes in the variables (prices and quantities) consequent on the tax are large, then there is no longer available the convenient generalisation of Cournot's formula for the change of price and quantity due to a tax on a monopolised article; the simultaneous equations of the first degree adduced in a former article (E). The neglect of the *third* powers of the increments which that method involves becomes inaccurate. The reasoning, too, may break down if the functions intended to represent the relation of price and quantity demanded are such and so simple that for the new values of the variables the functions no longer retain the character proper to the problem. This limitation is illustrated by the incident that the functions above employed to represent the relation of the price to the quantities x and y (above, p. 401) becomes inadmissible when x the number of first-class passengers is smaller than 85,000, while y the number of second-class passengers remains 200,000. For then $\frac{dp_1}{dx}$ is no longer, as it ought to be, *negative*.]

One more coefficient calls for one more remark: elasticity, in the popular sense, that is, $F'(p)$ in Cournot's notation, $\frac{dx}{dy}$ in ours. The sort of reader who is content with this usage may be apt to think that the distinction (*ante*, p. 290) which we have emphasised between elasticity and the increment thereof is a refinement of no great practical importance, that what is true of the increment is true enough of the quantity supposed to increase. It may be well, therefore, to point out that between the increment (first differential coefficient) of a variable and the variable itself, there may be all the difference that there is between the velocity at which a body is moving and the distance through which it has moved. Contrast the following propositions:—(1) The higher the speed of a motor-car the greater is the danger of accidents; (2) the longer the distance (from any fixed point) that a motor-car has travelled (at whatever rate), the greater is the danger of accidents. The former presumption could doubtless be verified by the statistics of accidents. Governments are well advised in making regulations based on this presumption. But what should we think of an expert who advised Government to discourage motorists from travelling beyond a certain distance from, say, New York, in order to prevent accidents? That advice would be of a piece with the theory which predicates of elasticity what is true of the increment of elasticity. No doubt it may be a proof of great natural ability to approach and half discern the truth in such a matter without the aid of the appropriate mathematical conceptions.